1. A steel beam of length $L$ with variable cross-section $A(x)$ is compressed by a force $P$ into a rigid wall, as shown in Figure 1, where $x$ is distance along the beam. The governing equation for the axial displacement of the beam is

$$A(x)E \frac{du}{dx} - P = 0, \quad (1)$$

subject to the boundary condition $u(0) = 0$. Here $E$ is the elastic modulus of the beam and the beam cross-sectional area varies linearly with distance as $A(x) = 1 - \varepsilon x$. Use the Galerkin finite element method to solve for $u(x)$ with the following expansion

$$u(x) = C_1 x + C_2 x^2 + C_3 x^3. \quad (2)$$

For this problem, we set $P/E = 0.001$ and $\varepsilon = 0.2$ in dimensionless units. Compare the finite-element solution for $u(1)$ with the exact solution.

Figure 1. A beam being compressed into a rigid wall.
2. We consider two-dimensional fluid flow past a circular cylinder, with radius \( a \) and flow speed \( U \) far away from the cylinder. In the absence of viscous friction, the fluid flow is governed by the Laplace equation

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \tag{2.1}
\]

where \( \phi(x, y) \) is related to the fluid velocity by \( \mathbf{u} = \nabla \phi \). The Laplace equation can be solved using a boundary-element method in terms of source panels on the cylinder surface \( C \). If the cylinder surface is discretized into \( N \) equal-length panels, each of strength \( q_i, i = 1, ..., N \), the boundary-element equation for the source strength is given by

\[
-U_{n_x} = \frac{1}{2} q(x) + \frac{1}{2\pi} \int_C q(x') \frac{\mathbf{n} \cdot \mathbf{r}}{r^2} \, dl', \tag{2.2}
\]

where \( \mathbf{n} \) is the outward unit normal of \( C \), \( \mathbf{r} = \mathbf{x} - \mathbf{x}' \), and \( r = |\mathbf{r}| \). Once the source strengths are obtained, the velocity \( \mathbf{u} \) at a location \( \mathbf{x} \) can be obtained by differentiating the potential function \( \phi(x, y) \) to yield

\[
\mathbf{u}(\mathbf{x}) = U \mathbf{e}_x + \frac{1}{2\pi} \int_C q(x') \frac{\mathbf{r}}{r^2} \, dl'. \tag{2.3}
\]

Discretizing the integral in (2.2) using the midpoint rule, we obtain a matrix equation for source strength \( q_n \) of the form

\[
\sum_{n=1}^N K_{mn} q_n = U \sin \theta_m, \tag{2.4}
\]

where \( \theta_m \) is the angle of the outward unit normal of the \( n^{th} \) surface panel (with \( \theta_n = 0 \) for a case with unit normal in the \( x \)-direction) and

\[
K_{mn} = \frac{1}{2\pi} \frac{(y_m - y_n) \cos \theta_m - (x_m - x_n) \sin \theta_m}{(x_m - x_n)^2 + (y_m - y_n)^2} \quad \text{for } m \neq n \tag{2.5a}
\]

and

\[
K_{mm} = -\frac{1}{2}. \tag{2.5b}
\]
Similarly discretizing (2.3) yields the velocity as a sum over the panel strengths as

\[
\mathbf{u}(x, y) = U\mathbf{e}_x + \frac{1}{2\pi} \sum_{n=1}^{N} \left[ \frac{(x-x_n)\mathbf{e}_x + (y-y_n)\mathbf{e}_y}{(x-x_n)^2 + (y-y_n)^2} \right] q_n, \tag{2.6}
\]

where \( \mathbf{e}_x \) and \( \mathbf{e}_y \) are unit vectors in the \( x \)- and \( y \)-directions.

Consider a special case with \( a = 1 \) and \( U = 1 \). Solve the matrix equation (2.4) for the panel source strengths using Matlab with \( N = 10 \) and 25 panels. Substitute your solution for the panel strength into (2.6) to obtain the velocity along a line \( y = 0 \) approaching the cylinder over the interval \(-10 < x < -1\) (indicated by a dashed line in Figure 2) for both \( N = 10 \) and 25, and evaluate your results by plotting the numerical solutions for these two cases versus the exact solution \( u = U(1 - a^2 / x^2) \).

![Figure 2](image_url)

**Figure 2.** Schematic of fluid flow past a circular cylinder, showing the line upstream of the cylinder along which you should plot the velocity variation with \( x \).

3. **Use a Lagrangian random-walk (stochastic) method to solve Problem 1 of Problem Set 3, involving advection and diffusion of a contaminant in a one-dimensional domain, for a case with diffusion coefficient \( \alpha = 2 \) and velocity \( u = 10 \). Employ the second-order predictor-corrector method to step the problem forward in time. Perform two sets of computations, one with time step \( \Delta t = 0.01 \) and one with \( \Delta t = 0.1 \). Plot the contaminant concentration field \( \phi(x, t) \) versus \( x \) at time \( t = 1 \) for both of these cases, and compare to the exact solution given in Problem Set 3.

4. **Solve the advection-diffusion equation using a spectral method for a case with periodic initial condition**

\[
\phi(x,0) = \cos x.
\]

The solution is assumed to remain periodic with period \( 2\pi \) for all time. Assume that the diffusion coefficient \( \alpha = 2 \) and the velocity \( u = 10 \). For this problem, the \( N = 2 \) term of the spectral expansion yields the exact solution.