1. Consider transport of a contaminant on an infinite one-dimensional domain. The contaminant concentration \( \phi(x,t) \) is initially given by

\[
\phi(x,0) = \exp(-x^2).
\]

a.) Suppose that the contaminant concentration field is advected by a uniform velocity \( u = 10 \) with no diffusion, such that

\[
\frac{\partial \phi}{\partial t} + u \frac{\partial \phi}{\partial x} = 0.
\]

Plot \( \phi(x,t) \) versus \( x \) at times \( t = 0, 0.5, \) and \( 1.0. \)

b.) Suppose that the contaminant concentration field diffuses outward with diffusion coefficient \( \alpha = 2 \), such that

\[
\frac{\partial \phi}{\partial t} = \alpha \frac{\partial^2 \phi}{\partial x^2}.
\]

This equation admits a solution of the form

\[
\phi(x,t) = \frac{1}{[4\alpha(t + c)]^{1/2}} \exp\left[-\frac{x^2}{4\alpha(t + c)}\right],
\]

where \( c = 1/4\alpha \). Plot \( \phi(x,t) \) versus \( x \) at times \( t = 0, 0.1, \) and \( 0.2. \)

The advection-diffusion equation contains both the advective and diffusive behaviors observed in the two problems above.
2. Determine whether each of the following partial-differential equations is elliptic, parabolic, or hyperbolic. State the boundary and initial conditions of a well-posed problem for this equation.

a.) Equation for waves on a string under tension:

\[ \frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2} - a \frac{\partial \phi}{\partial x}, \]

where \( \phi(x, t) \) is the string displacement, \( x \) is distance along the string, \( c \) is the wave speed, and \( a \) is a wave dissipation coefficient.

b.) Equation for steady-state heat conduction in a material with an exothermic reaction providing an energy source, occupying a two-dimensional region A:

\[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = f(x, y), \]

where \( \phi(x, t) \) is temperature.

c.) Equation for advection and diffusion of a contaminant in a small river

\[ \frac{\partial \phi}{\partial t} + u \frac{\partial \phi}{\partial x} = \alpha \frac{\partial^2 \phi}{\partial x^2}, \]

where \( \phi(x, t) \) is contaminant concentration, \( x \) is distance along the river, \( u \) is mean river velocity, and \( \alpha \) is diffusion coefficient (\( \alpha > 0 \)).

**For Problems 3-4.** Consider the problem of bending of a beam that is pinned at both ends and subject to a combination of longitudinal and transverse loads, as shown below. The beam transverse (y) displacement \( \phi(x) \), which is assumed to be small compared to the beam length, is governed by the equation

\[ EI \frac{\partial^4 \phi}{\partial x^4} + P \frac{\partial^2 \phi}{\partial x^2} = q(x), \]

(1)

where \( EI \) is the bending stiffness, \( x \) is distance along the beam, \( P \) is the applied compressive force at the ends, and \( q(x) \) is the transverse loading per unit length. The dashed line in the figure below is a slot along which the pins at the beam ends are free to travel. At the ends of the beam (\( x = 0 \) and 1), the boundary conditions require zero transverse displacement.
\[ \phi(0) = 0, \quad \phi(1) = 0, \]  

and zero bending moment

\[ \frac{\partial^2 \phi}{\partial x^2}(0) = 0, \quad \frac{\partial^2 \phi}{\partial x^2}(1) = 0. \]  

For Problems 3-4, we consider a case with the transverse loading is uniform along the beam, or \( q(x) = q_0. \)

3. Solve the problem exactly for a case with \( EI = 1, \ P = 1, \) and \( q_0 = 5, \) in dimensionless units, and determine the maximum deflection of the beam (at \( x = 0.5 \)).

4. Use the finite-difference method to solve for the rod deflection \( \phi(x) \). Start by writing the discretized form of the equation using central differencing. Next, put the discretized equation in the form of a matrix equation

\[ \sum_{j=1}^{N} M_{ij} \phi_j = f_i, \quad i = 1, \ldots, N. \]  

Be careful to account for the boundary conditions in setting up the matrix. Then solve this matrix equation using Matlab. Make a plot of \( \phi(x) \) versus \( x \) for cases with \( N = 5, 10, \) and \( 20 \) and compare with the exact solution. Make a table comparing the maximum deflection of the beam with the exact solution for these cases. SHOW ALL OF YOUR WORK!
5. Consider diffusion of chlorine in a square, shallow swimming pool, which is assumed to be well-mixed in the vertical direction. The chlorine concentration \( \phi(x,y,t) \) is governed by

\[
\frac{\partial \phi}{\partial t} = \alpha \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right),
\]

where \( \alpha \) is a diffusion coefficient and \( x \) and \( y \) are horizontal directions. Without the pool filter running, the boundary conditions are that the gradient of \( \phi \) normal to the boundary (i.e., the pool side) must vanish.

divide the pool into a Cartesian grid in the horizontal directions, denoting each grid cell by \( V_i \). Integrate (6) over \( V_i \) and employ the divergence theorem to obtain the finite-volume form of the equation in one grid cell.

b.) Prove that if this problem is solved using the finite-volume method, the total amount of chlorine in the pool will remain constant in time. (A solver that has this property is said to be conservative.)

6. Derive the numerical stability condition for a finite-difference solution of the heat equation,

\[
\frac{\partial \phi}{\partial t} = \alpha \frac{\partial^2 \phi}{\partial x^2},
\]

using backwards differencing in time and centered differencing in space.