Let's assume we have a 3x3 neighborhood median filter, positioned at some position p.

Image $A = \begin{bmatrix} a & a & a \\ a & b & b \\ b & b & b \end{bmatrix}$ with $a > b$ 

Image $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 0 \end{bmatrix}$ with $c = b - a$ 

Median of $A$ is "$a$". 
Median of $B$ is "$b$". 

$A + B = \begin{bmatrix} a & a & a \\ a & b & b \\ b & b & b \end{bmatrix}$ 

Median of $A + B$ is "$b$".

Since median $A + \text{median } B \neq \text{median } (A + B)$, the median operator is not linear.
Given $P_v(r)$, transform to new $P_v(z)$.

Follow method from Eq. 3.3.2.

\[ T(r) = \int_0^r P_v(w) \, dw = \int_0^r (2 - 2w) \, dw \]

\[ = 2w - \frac{2w^2}{2} \bigg|_0^r = 2r - r^2 \]

\[ G(z) = \int_0^z P_v(t) \, dt = \int_0^z 2t \, dt \]

\[ = \frac{2t^2}{2} \bigg|_0^z = z^2 \]

So, from Eq. 3.3-12, the transformation is:

\[ z = G^{-1}(T(r)) = \sqrt{2r - r^2} \]
3.17 Box filter vs. Brute Force

Brute force requires $n \times n - 1$ adds/pixel, for a total of $(n^2 - 1) \times N \times M$ operations.

Box filter requires $n^2 - 1$ adds for first location, but only $n+1$ adds for each location after that.

$$\begin{array}{c|c|c|c} \hline C_0 & C_1 & C_n \hline \vdots & \vdots & \vdots \hline \end{array}$$
\text{add new column}

$$\begin{array}{c|c|c|c} \hline C_0 & C_1 & C_n \hline \vdots & \vdots & \vdots \hline \end{array}$$
\text{subtract old column}

Total is $(n^2 - 1) \times (n+1) \times (N \times M - 1)$

Computation advantage is $\frac{n^2 - 1}{n+1} = \frac{n-1}{1}$
\[
\mathcal{M}(x, \omega) = \frac{1}{16} \left[ \begin{array}{c}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2}
\end{array} \right] e^{-j(\omega, x + \Omega z \gamma)}
\]

\[
\mathcal{H}(x_1, \omega_2) = \sum_{\gamma} \sum_{\mathcal{E}} h(\nu, x) e^{-j(\nu, x)}
\]

\[
= \left( 4 + 2 e^{-j\nu_1} + 2 e^{j\nu_2} \right)
\]

\[
\left( \begin{array}{c}
\nu_1 - j\nu_2 \\
\nu_1 + j\nu_2
\end{array} \right)
\]

\[
\left( \begin{array}{c}
\nu_1 - j\nu_2 \\
\nu_1 + j\nu_2
\end{array} \right)
\]

\[
\left( \begin{array}{c}
\nu_1 - j\nu_2 \\
\nu_1 + j\nu_2
\end{array} \right)
\]

\[
\left( \begin{array}{c}
\nu_1 - j\nu_2 \\
\nu_1 + j\nu_2
\end{array} \right)
\]

\[
= \frac{1}{16} \left[ \begin{array}{c}
4 + 4 e^{j\omega_1 \Omega x} - 4 e^{j\omega_1 \Omega x}
\end{array} \right]
\]
b) For \( \Omega_2 = 0 \),

\[
H(a_0, 0) = \frac{1}{16} \left[ 1 + 4 \cos \Omega_1 \cdot 4 + 4 \cos \Omega_1 \right]
\]

\[
= \frac{1}{16} \left[ 8 + 8 \cos \Omega_1 \right]
\]

\[
= \frac{1}{2} \left[ 1 + \cos \Omega_1 \right]
\]

Compare this to 3x3 averaging mask,

\[
H(a_1, \Omega_1) = \frac{1}{3} \left( 1 + 2 \cos \Omega_1 \right)
\]

The weighted mask has broader passband, but better high freq. attenuation.

See attached plot.