the accuracy of the method can be found from Hong et al. (1994a), Hong and Rubinsky (1994), and Hong (1993).

Conclusion

A new technique for analyzing problems of heat transfer with phase transformation in multiple domains was described. The technique employs interactive MRI imaging, thermocouple measurements, and finite difference analysis. The paper illustrates the ability of the technique to calculate the temperature distribution in multiple freezing domains that merge. MRI interactive heat transfer analysis may become an important tool in the armamentarium of researchers in heat transfer.

Acknowledgments

This research was supported by NIH grant No. RO1 CA56898-03A1.

References


Boundary Layer Analysis of Buoyancy-Driven Two-Phase Flow in Capillary Porous Media

C. Y. Wang1 and C. Beckermann2

Nomenclature

\[ D = \text{dimensionless capillary diffusion function} = k_c(1 - \lambda)[{-dJ(s)/ds}] \]

\[ D_y = \text{constant part of dimensionless capillary diffusion function} = \frac{(cK)^2/\sigma}{\nu} \]

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2 Associate Professor, Department of Mechanical Engineering, The University of Iowa, Iowa City, IA 52242; Mena, ASME. Contributed by the Heat Transfer Division of the AMERICAN SOCIETY OF MECHANICAL ENGINEERS. Manuscript received by the Heat Transfer Division September 1993; revision received January 1995. Keywords: Multiphase Flows, Phase-Change Phenomena, Porous Media. Associate Technical Editor: C. E. Hickox, Jr.

1 Introduction

Buoyancy-driven two-phase flow in a porous medium is encountered in numerous important technological applications. Examples include boiling flow along an igneous intrusion in geothermal reservoirs and condensing flow adjacent to a cold surface in heat pipes and porous insulation materials. Recently, condensing flow has received particular attention, because the flow in the two-phase region, which results from the capillary force, is believed to exert a significant effect on film condensation heat transfer (Plumb, 1984; Shekarziz and Plumb, 1986; Majumdar and Tien, 1990; Plumb et al., 1990; Chung et al., 1992). An analogous effect can be expected for film boiling in porous media, although no study has been conducted.

Prior work (Plumb, 1984; Shekarziz and Plumb, 1986; Majumdar and Tien, 1990) has shown that the two-phase flow within the above context exhibits features similar to boundary layer flow of single-phase fluids (Schlichting, 1968). Based on the separate flow model, these theoretical studies, however, neglect the vapor flow in order to render the problem analytically tractable. More specifically, the vapor phase is assumed to remain at a constant pressure; a two-phase flow problem is thus essentially reduced to the consideration of the liquid phase flow only. This approach, known as the unsaturated flow theory, is widely applied in hydrology (Morel-Seytoux, 1973), and the corresponding assumption is called the Richards approximation. While the theory has been shown to be a good approximation...
in many hydrological situations, doubts remain regarding its validity in two-phase flow with phase change.

The goal of the present paper is twofold. First, based on a newly developed two-phase mixture model (Wang and Beckermann, 1993a), we perform a two-phase boundary layer analysis for buoyancy-driven two-phase (both condensing and boiling) flow in porous media, wherein only classical boundary layer approximations (Schlichting, 1968) are employed. A parallel theory for pressure-driven boiling flow has been presented by Wang and Beckermann (1993b). Second, using the present full two-phase solution, we quantitatively examine the validity of the unsaturated flow theory for buoyancy-driven condensing flow, so as to reveal the capillary effect on film condensation (or boiling) heat transfer in porous media more accurately.

2 Boundary Layer Analysis

Both boiling and condensing flows are dealt with in the present paper, as shown in Figs. 1(a) and 1(b). In each case, a two-phase zone where the liquid and vapor phases can coexist appears adjacent to the vertical wall. As the heat transfer at the wall is gradually enhanced, the liquid saturation at the wall approaches its respective limiting value, namely zero for boiling and unity for condensation. If the heat flux further exceeds this threshold point, a pure vapor or liquid film forms at the wall. A complete problem then involves multiple regions consisting of a single-phase film on the wall and the adjacent two-phase zone. The analysis in this paper focuses solely on the two-phase flow without a single-phase film covering the wall, followed by a brief discussion on its extension to a multiregion problem.

Like in all previous theoretical work, use is made of common assumptions, such as constant thermophysical properties, an isothermal two-phase zone, and the Darcian flow regime for both phases. However, simultaneous two-phase flow is considered in the following analysis, and hence the two-phase characteristics of the problem are fully retained.

2.1 Two-Phase Boundary Layer Equations. The present analysis is based on the two-phase mixture model recently proposed by the present authors (Wang and Beckermann, 1993a). The model is a mathematically equivalent but reformulated version of the classical separate flow model. The central idea underlying the approach is to view the two-phase system similar to a binary chemical mixture. Hence, a two-phase flow can uniquely be described by a mass-averaged mixture velocity, \( u \), and a diffusive flux, \( j \), which is defined as the difference between the mixture velocity and an individual phase velocity, \( u_l \) or \( u_v \). It has been shown in Wang and Beckermann (1993a) that the diffusive mass flux, \( j \), defined as

\[
 j = \rho u_l - \lambda \rho u = -[\rho u_l - (1 - \lambda) \rho u]
\]

is given by

\[
 j = -D(s) \nabla s + f(s) \frac{K \Delta \rho}{\nu_e} g
\]

where \( D(s) \) and \( f(s) \) are the capillary diffusion coefficient and hindrance function, respectively, as defined in the Nomenclature. Throughout this paper, symbols without a subscript are reserved for mixture variables, while the subscripts \( v \) and \( l \) denote variables of the vapor and liquid phases, respectively. It can be seen from Eq. (2) that the diffusive mass flux is due to a combination of capillary diffusion and gravity-induced migration.

A set of boundary-layer equations has been derived by Wang and Beckermann (1993b) for the two-phase mixture motion. In the Cartesian coordinate systems shown in Fig. 1, the mixture mass conservation, mixture Darcy’s law, and liquid mass conservation equations can be written as

\[
 \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0
\]

\[
 \rho u = -K \frac{\partial}{\partial x} \left( \frac{\partial \rho}{\partial x} \pm \rho g \right)
\]

\[
 \rho \frac{d \lambda}{ds} \left( u \frac{\partial s}{\partial x} + v \frac{\partial s}{\partial y} \right) = \frac{\partial}{\partial y} \left( D \frac{\partial s}{\partial y} \right) \mp \frac{K \Delta \rho}{\nu_e} g \frac{df}{ds} \frac{\partial s}{\partial y}
\]

The boundary conditions for Eqs. (3)–(5) are

\[
 \text{at } x = 0, \quad u = 0 \text{ and } s = \begin{cases} 1 & \text{boiling flow} \\ 0 & \text{condensing flow} \end{cases}
\]

\[
 \text{at } y = 0, \quad v = 0 \text{ and } s = s_w
\]

and

\[
 \text{when } y \to \infty, \quad u = 0 \text{ and } s = \begin{cases} 1 & \text{boiling flow} \\ 0 & \text{condensing flow} \end{cases}
\]

where \( s_w \) is the constant liquid saturation at the wall, with \( s_w = 1 \) representing the marginal condition for a pure liquid film to appear on the wall in the condensing case, and \( s_w = 0 \) corresponding to the dryout condition in the boiling case. The more general case of a saturation that varies along the wall has been examined by Chung et al. (1992). The present use of a uniform wall saturation enables a similarity solution to be obtained and a direct comparison with the results of Majumdar and Tien (1990) (see below).

Finally, the basic transport functions \( k_u(s), k_{nw}(s) \), and \( J(s) \) must be provided to determine the mixture mean transport properties (see Nomenclature). The following relative permeabilities and capillary pressure function are assumed in the present study:

\[
 k_u = s^3; \quad k_{nw} = (1 - s)^3
\]

and

\[
 J(s) = 1.417(1 - s) - 2.120(1 - s)^2 + 1.263(1 - s)^3
\]

where the irreducible liquid saturation has been taken to be zero for simplicity. This set of constitutive relations has been used in several previous studies (Shekarriz and Plumb, 1986; Ma-
2.2 Comparisons With Unsaturated Flow Theory. The
unsaturated flow theory is valid in the limiting case where the
vapor pressure can be assumed to be constant. Its parametric
range can be revealed by a simple order-of-magnitude analysis.
By application of the modified Darcy’s law for two-phase flow,
and taking the relative permeabilities to be of the order of unity,
the ratio of the pressure drops across the two-phase zone can
be approximated by

\[ \frac{\Delta p_2}{\Delta p_1} \sim \frac{\mu_2}{\mu_1} \frac{\nu_2}{\nu_1} \frac{\rho v_2}{\rho v_1} \]  

(11)

Noting that the normal mixture velocity vanishes at the wall
[Eq. (7)], the normal mass fluxes of the liquid and vapor phases
be assumed to be of the same order of magnitude (but opposite in sign). Then, Eq. (11) reduces to

\[ \frac{\Delta p_2}{\Delta p_1} \sim \frac{\nu_2}{\nu_1} = 1/\varphi \]  

(12)

where \( \varphi \) is the ratio of the kinematic viscosities of the liquid to
vapor. It can then be concluded that, for \( \Delta p_2 \), to be negligibly
small as required in the unsaturated flow theory, \( \varphi \) must approach infinity. In practice, however, \( \varphi \) can be very small (e.g., \( \varphi = 0.01466 \) for a water-steam system at
atmospheric pressure). This points out the possibility that the
vapor phase may significantly retard the movement of the liquid
phase, so that a full two-phase model must be adopted. Indeed,
the most recent work by Chung et al. (1992) has realized this
need and attempted to account for the vapor flow.

In the limiting case where \( \varphi \to \infty \), it can be shown that

\[ \frac{d\eta}{ds} \approx O(\varphi^{-1}) \to 0; \quad D/D_e \to k_n(s)[-J'(s)]; \]

\[ \frac{1}{\nu_2} \frac{d\eta}{ds} \to \frac{1}{\nu_1} \frac{d\eta}{ds} \]  

(13)

Then, the governing equation for condensing flow, Eq. (5),
reduces to

\[ \frac{D}{D_e} \frac{\partial}{\partial y} \left[ k_n(s)[-J'(s)] \frac{\partial \eta}{\partial y} \right] - \frac{K \Delta p_2}{\nu_1} \frac{\partial k_2}{\partial s} \frac{\partial \eta}{\partial x} = 0 \]  

(14)

This is exactly the key equation solved by Majumdar and Tien
(1990) in their analysis using the unsaturated flow theory.
Hence, the foregoing order-of-magnitude analysis is substi-
tuated.

2.3 Similarity Solution. Equations (3)–(5) along with
their boundary conditions admit a similarity solution by virtue
of the following similarity variables:

\[ \eta = \frac{y}{x} \sqrt{\alpha}; \quad \Psi = D_2 \alpha_2^{1/2} F(\eta) \quad \text{and} \quad s = s(\eta) \]  

(15)

where the local two-phase Rayleigh number, \( \alpha_2 \), is defined in
the nomenclature, and \( \Psi \) is a streamfunction of the two-phase
mixture such that \( \mu u = \partial \Psi/\partial y \) and \( \mu v = -\partial \Psi/\partial x \). This similarity
transformation reduces Eqs. (3)–(5) to the following set of
ordinary differential equations:

\[ F' = \left\{ \begin{array}{ll}
\frac{d k_n}{k_n} & \text{boiling flow} \\
\eta s' & \text{condensing flow}
\end{array} \right. \]  

(16)

\[ (Ds')' + \frac{1}{2} \frac{d x}{d s} Fx' = \pm \frac{1}{2} \varphi \frac{d f}{d s} \eta s' \]  

(17)

The corresponding boundary conditions are

\[ \text{at } \eta = 0, \quad F = 0 \quad \text{and} \quad s = s_a \]  

(18)

when \( \eta \to \infty \) and

\[ \begin{array}{ll}
F' = 0 & \text{boiling flow} \\
0 & \text{condensing flow}
\end{array} \]  

(19)

In the equations above the prime denotes the derivative with
respect to \( \eta \).

Equations (16) and (17) along with their boundary conditions
(18) and (19) were numerically solved by the Gear stiff
method (IMSL, 1991) combined with a shooting procedure
for \( s'(0) \). The singularity at \( \eta = 0 \), when the wall saturation
approaches zero in the boiling case and unity in the condensing
case, was treated in the same manner as in Wang and Becke-
mann (1993b). The present numerical procedures were vali-
dated by comparing the saturation profile obtained for the lim-
iting case \( \varphi \to \infty \) with that of Majumdar and Tien (1990), and
excellent agreement was achieved.

The two-phase Sherwood number, defined again in the No-
mencature, stands for the dimensionless phase change rate at
the wall. By the definition of the diffusive mass flux, Eq. (2),
and using the similarity transformation, one can obtain

\[ \frac{Sh}{Ra_2^{1/2}} = -\frac{D(x_0)}{D_e} \]  

(20)

The Sherwood number is also a direct measure of the heat flux
imposed at the wall, such that

\[ \frac{q_{w,x}}{h_0 D_e} = -Sh = \frac{D(x_0)}{D_e} \sqrt{Ra_2} \]  

(21)

where \( q_w \) denotes the wall heat flux. Integration of the equation
above yields the average heat flux over the entire wall:

\[ \frac{q_{w,L}}{h_0 D_e} = 2\frac{D(x_0)}{D_e} \sqrt{Ra_2} \]  

(22)

which can be used to estimate the dryout heat flux in the boiling
case when \( s_a = 0 \). In the condensing case, on the other hand, the
liquid mass flux as calculated from the Sherwood number will
reflect the capillary effect on film condensation heat transfer.

Having found the mixture velocity field from the similarity
solution, the individual phase velocities can be calculated from
Eq. (1) combined with Eq. (2). An important characteristic of
the phase velocity fields can readily be inferred from the govern-

ing equations, Eqs. (16) and (17), by substituting the definition
of the hindrance function given in the nomenclature. This leads to

\[ [D s' - \frac{1}{2}(\eta F' - F) s'] = 0, \quad \text{boiling flow} \]  

\[ [D s' + \frac{1}{2}(1 - \lambda)(\eta F' - F)] s' = 0, \quad \text{condensing flow} \]  

(23)

(24)

Since the terms in parentheses actually represent the transverse
components of the liquid and vapor velocities in boiling and
condensing flows, respectively, one can conclude that the liquid
(vapor) flow is exactly one-dimensional in the direction normal
to the wall in boiling (condensing) flow. Physically, this is due
to the fact that there is no driving force for liquid (vapor)
motion in the longitudinal direction in the respective situation.
For example, in boiling flow, the liquid momentum equation in
the x direction reads

\[ u_i = -\frac{K k_n}{\mu_i} \left( \frac{d p_1}{d x} - \rho g \right) \]  

(25)

where the bracket on the right-hand side remains constant along
the y direction, as implied by the boundary-layer approxima-

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3 Results and Discussion

Numerical results are presented below separately for boiling and condensing flows. Inspection of the similarity equations reveals that the solution depends only on the viscosity ratio, \( \nu \), and the wall liquid saturation, \( s_w \). Although the density difference between the two phases also has a substantial influence on the behavior of these types of flow, it does not enter the present similarity solution because of the definition of the mixture streamfunction, \( \Psi \).

3.1 Boiling Flow. Illustrative calculations for boiling flow are performed for a water–steam system at atmospheric pressure. Hence, the value of \( \nu \) is fixed at 0.01466. Focus is placed on examining the effect of the wall liquid saturation (i.e., the wall heat flux) on two-phase flow behaviors as well as determining the dryout heat flux for boiling along a vertical plate in a porous medium.

Figure 2(a) shows the liquid saturation profiles with the wall saturation as the curve parameter. It can be noted that these profiles are complex in shape and possess several points of inflection, contrasting with the simple concentration profile in single-phase mass transfer problems. This feature results from the fact that the mean transport properties of a two-phase mixture are highly nonlinear and exhibit maxima at certain liquid saturations. The second distinctive feature of the saturation profiles shown in Fig. 2(a) is the sharp edges of two-phase boundary layers for all wall liquid saturations, since the capillary diffusion coefficient becomes extremely small.

The quantity \( \text{Sh} / \text{Ra}^{1/2} \) is shown in Fig. 2(b) as a function of the wall liquid saturation. It is noted that it increases as the wall saturation decreases and finally converges to a value of \( 1.358 \times 10^{-3} \) for the dryout condition \( (s_w = 0) \). Hence, the dryout heat flux is given by

\[
\frac{q_w L}{h_f D_c} = 2.716 \times 10^{-3} \text{ Ra}^{1/2}
\] (26)
This result together with those obtained in a previous study on pressure-driven flow (Wang and Beckermann, 1993b) represent the useful bounds for boundary-layer boiling flows.

The axial mixture velocity, \( \mu u \) (i.e., \( F' \)), can readily be evaluated from the saturation profiles via Eq. (16). Figures 2(c) and 2(d) instead depict the liquid and vapor phase velocity fields, respectively, in a typical two-phase boundary layer when \( s_w = 0 \) and \( Ra_L = 50 \). The dashed line in these plots represents the edge of the boundary layer. The vapor moves primarily vertically due to the large density difference between liquid and vapor. The liquid is laterally entrained into the boundary layer, as indicated theoretically by Eq. (23). It is noteworthy that the entrainment is due to buoyancy-induced flow near the wall in the present problem, while the corresponding lateral motion in pressure-driven flow is caused by viscous displacement (Wang and Beckermann, 1993b).

### 3.2 Condensing Flow

The main objective of this subsection is to compare the present full two-phase solution with the approximate solution obtained by Majumdar and Tien (1990) using the unsaturated flow theory. The wall liquid saturation is chosen as unity (i.e., the marginal condition for a liquid film to appear). Hence, the viscosity ratio becomes the sole parameter. Figure 3(a) shows the liquid saturation profiles for different values of the viscosity ratio along with the curve obtained using the unsaturated flow theory (Majumdar and Tien, 1990). The latter is independent of the viscosity ratio \( \nu \). It can be seen that the curves indeed approach the limiting case represented by the unsaturated flow theory. However, the value of \( \nu \) for the unsaturated flow theory to hold is surprisingly large, i.e., \( \nu = 10^7 \). Obviously, for all common fluids, this condition cannot be satisfied, indicating that the unsaturated flow theory is questionable when applied to the condensation problem under consideration. In fact, for a water-steam system at atmospheric pressure \( (\nu = 0.01466) \), the saturation profile from the two-phase model greatly deviates from that predicted by the unsaturated flow theory.

As mentioned earlier, the capillary effect on condensation heat transfer is reflected by the liquid mass flux at the wall, \( j_w \), or \( Sh^* / Ra_L^{1/2} \) in a dimensionless form. Figure 3(b) depicts the influence of \( \nu \) on \( Sh^* / Ra_L^{1/2} \). It is clearly seen that the unsaturated flow theory greatly overpredicts the dimensionless liquid mass flux, with \( Sh^* / Ra_L^{1/2} = 0.288 \), while the present two-phase model gives a value of only \( 4.35 \times 10^{-3} \). This 66-fold differ-
ence suggests that the previous theoretical investigations based on the unsaturated flow theory (Plumb, 1984; Shekarriz and Plumb, 1986; Majumdar and Tien, 1990) have exaggerated the effect of the capillary force on condensation heat transfer in porous media. The overprediction results from the neglect of the vapor flow that is actually very strong due to the low vapor density.

Figures 3(c) and 3(d) display the liquid and vapor flow fields in the two-phase boundary layer corresponding to a wall saturation of unity. The dashed line in these plots again denotes the edge of the boundary layer. Also note the large difference in the velocity scale given at the top of the figures. The liquid flow is essentially confined to a region very near to the wall where the vapor condenses into the liquid. As a result of its large density, the liquid moves primarily downward (however, the normal liquid velocity is not zero). The axial liquid velocity quickly diminishes with the normal distance, because it is proportional to the cubic of the liquid saturation, which furthermore has an extremely steep profile [see Fig. 3(a)]. Figure 3(d) shows that the vapor is only laterally entrained into the two-phase boundary layer, confirming the theoretical outcome of Eq. (24).

The present result agrees well with the assumption of Chung et al. (1992) regarding one-dimensional vapor flow. However, it should be kept in mind that the result is a consequence of the present analysis. Therefore, the theoretical framework itself is not subject to modifications in general situations where two-dimensional vapor flow is present, for example, in combined buoyancy- and pressure-driven two-phase flow.

Unfortunately, the available condensation experiments (White and Tien, 1987; Chung et al., 1992) appear to be inappropriate for validation of the present model. As pointed out by Kaviany (1991), the measured liquid film thicknesses in these experiments are of the same order as the bead diameters, causing a continuum treatment like the Darcy's formulation to break down. Furthermore, the experiments seem to be not representative of the capillary effect on condensation heat transfer. Indeed, as indicated by Plumb (1990), the physical models of Cheng (1981) and Chung et al. (1992) produce very similar results for the condensation rate under these experimental conditions, even though the latter is much more sophisticated and includes many non-Darcian effects as well as the capillary effect.

### 3.3 Combined Single- and Two-Phase Flow

Beyond the marginal condition where either \( s_w = 0 \) in boiling flow or \( s_w = 1 \) in condensing flow, a single-phase film appears on the wall in addition to the two-phase zone. In this case, the two-phase analysis needs to be extended to incorporate the single-phase flow within the film, and the latter can readily be dealt with, as done in many previous studies. The two-phase analysis is still valid except that the phase interface becomes permeable. As a result, the normal mixture velocity, \( u_n \), is no longer zero as in Eq. (7), but must be obtained from the coupling between the single- and two-phase flows. Such an extension is not performed here for brevity.

### 4 Conclusions

Complementing a previous investigation on pressure-driven flow (Wang and Beckermann, 1993b), the present study analytically explores buoyancy-driven two-phase flow along a vertical plate in a capillary porous medium. Full two-phase similarity solutions were obtained for both boiling and condensing flows. Illustrative numerical results were presented for the liquid saturation profile, the flow fields of the two phases within the two-phase boundary layer, and the phase-change rate at the wall. The dryout heat flux in boiling flow was correlated with the fluid thermophysical properties and porous medium parameters. Experimental verification of these results is, however, necessary. For condensing flow, the present work improves previous approximate solutions based on the unsaturated flow theory, and further indicates that the theory is questionable for condensing flow. Accurate modeling of the capillary effect on film condensation heat transfer appears to require a full two-phase treatment. Further experimental work should also concentrate on this effect.

A comparison of the boiling and condensing boundary layer flows, as shown in Figs. 2(c, d) and 3(c, d), indicates that the boiling boundary layer is normally wider than the one in condensing flow, and that a two-phase crossflow pattern is representative of both boiling and condensing buoyancy-driven flows.

### Acknowledgments

This work was supported by the National Science Foundation under Grant No. CTS-8957149 and by a University of Iowa Carver Scientific Research Initiative Grant.

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