ON THE DETERMINATION OF BOUNDARIES TO MANIPULATOR WORKSPACES

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The accumulation of two independent, broadly applicable formulations for determining the boundary to manipulator workspaces, presented elsewhere, are compared in this paper. Insights gained from one method are used to explain behavior exhibited in the other. Results are also compared and validated. A numerical formulation based on continuation methods is used to map curves that are on the boundary of a manipulator workspace. Analytical criteria based on row rank deficiency criteria of the manipulator's analytical Jacobian are used to map a family of one-dimensional solution curves on the boundary. The other formulation, based on a similar rank-deficiency criteria, yields analytic boundaries parametrized in terms of surface patches on the boundary. Results concerning the applicability of the numerical method to open- and closed-loop systems are compared with those limited to the open-loop for the analytical method. Conclusions regarding the behavior of the manipulator on geometric entities characterized by singular curves, higher-order bifurcation points, and surfaces inside the workspace are drawn. Applicability of both methods and their limitations are also addressed. © 1997 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

There has been considerable effort in recent years towards the formulation of mathematical methods for determining workspaces of mechanical systems. The motivation behind these formulations stems from applications that require mechanisms to be inserted into smaller spaces. The study of manipulator workspaces has also been identified in the fields of manufacturing for the efficient placement of robots on the shop floor and for securing the maximum functionality of a manipulator in terms of dexterity. Other applications of such studies have emanated in the medical field, where the use of mechanisms and machines in medical interventions has become inevitable.

This paper presents the culmination of two works that have appeared in the literature throughout recent years. The aim of this paper is not only to collect the state of the art in the field of workspace analysis and to present major results and contributions, but to explain the physical significance of earlier results that were not adequately addressed.

The first is based upon a numerical method whereby curves on the boundary of manipulator workspaces are mapped using a continuation method. Criteria for establishing a boundary were first presented by Jo and Haug,\cite{1,2} Haug et al.,\cite{3} and Wang and Wu.\cite{4} In these presentations, planar implementations of open- and closed-loop systems were addressed. Subsequently, Haug et al.\cite{5} presented the first complete treatment of the determination of curves to the boundary of manipulator workspaces including spatial mechanisms. Interior curves and surfaces were not, however, adequately addressed until a more recent presentation by Haug et al.\cite{6} Other aspects of manipulator functionality such as dexterous workspaces and difficulties encountered in computing bifurcation points on continuum curves were addressed by Qiu et al.\cite{7} The appearance of barriers inside the workspace that are impediments to motion were addressed by Haug et al.\cite{8}

The second method is based upon similar criteria for the rank deficiency of the analytic Jacobian, where internal and boundary singularities are characterized in terms of the Jacobian and the reduced-order manipulator.\cite{9,10} Singular sets upon which the manipulator loses one or more degrees of mobility are analytically determined and used to parametrize surface patches on the boundary to the workspace. Elementary results were first presented by Abdel-Malek and Yeh\cite{11,12} and Abdel-Malek. A more generalized formulation for serial manipulators was presented by Abdel-Malek and Yeh. More recently, results concerning the existence of singular surfaces inside the workspace, their crossability based on
normal acceleration analysis, and the behavior of the manipulator following a trajectory in the workspace were presented by Yeh and Abdel-Malek.\textsuperscript{33,34}

Prior to the works mentioned, there have been many works that have dealt with the general subject of manipulator workspaces. The reciprocal screw method for workspace generation, for example, is based on the fact that when the end-effector reference point of the manipulator is on the workspace boundary, all the joint axes of a manipulator are reciprocal to a zero pitch wrench axis.\textsuperscript{35} For each degree-of-freedom lost, there exists one reciprocal screw which, if applied as a wrench to the end-effector, produces no virtual work for the manipulator joints. Wang and Waldron,\textsuperscript{36} based upon earlier work,\textsuperscript{37} noted that the Jacobian of the manipulator becomes singular if its columns, which are screw quantities, do not span the full rank of the matrix, therefore reducing the Jacobian rank by at least one.

Other methods that are based on a Jacobian's singularity can be found in Refs 17, 18 and 24. An enumeration of singular configurations due to the vanishing of the determinant of the Jacobian and the Jacobian's minors is presented by Lipkin and Pohl.\textsuperscript{16} Shamir\textsuperscript{38} provided an analytical tool to determine whether the singularities are avoidable for three degrees-of-freedom manipulators. Govila\textsuperscript{8} reported obtaining expressions for the set of singular points by assuming that link twists were multiple of \(\pi/2\). Geometric approaches to the study of singular configurations of a manipulator arm were addressed by Lai and Yang,\textsuperscript{15} Ahmad and Luo,\textsuperscript{3} and Toussaint and Ang.\textsuperscript{27} Other earlier important studies that discussed manipulator singularities include Soylu and Duffy\textsuperscript{39} and Lai and Yang.\textsuperscript{14}

Early studies that have addressed difficulties in the control of manipulators due to the appearance of interior curves and surfaces were reported by Waldron\textsuperscript{28} and Nielsen et al.\textsuperscript{19} In the latter work, the controllability of a mechanical arm is discussed from a differential geometry point of view. Nielsen et al. discussed control difficulties based upon vector fields and their consecutive Lie brackets spanning a state space. The fundamental concept of crossable and non-crossable surfaces inside a manipulator's workspace was addressed by Oblas and Kohi.\textsuperscript{20}

In this paper, difficulties encountered in one method are analytically interpreted using the other. Validations of the underlying theorems presented by both methods are carried out by performing the analysis on a spatial manipulator with four degrees of freedom (RPRP). Results previously unexplained in both methods are now construed and characterized by the limitations imposed on both methods.

A summary of the underlying theory for both formulations will first be presented. An example will be implemented using both methods. Difficulties and limitations encountered by both methods will then be addressed.

2. TWO FORMULATIONS

Both formulations are briefly summarized in this section. Note that the first method yields curves that are boundary to the workspace while the second method yields surface patches on the boundary.

2.1. Method I: boundary curves to the workspace\textsuperscript{10}

In a neighborhood of an assembled configuration \(\mathbf{q}\) of a mechanism, generalized coordinates satisfy independent holonomic kinematic constraint equations of the form

\[ \Phi(q) = 0 \]  

where \(q = [q_1, q_2, \ldots, q_N] \in \mathbb{R}^N\) is the vector of joint coordinates. Output coordinates of the mechanism are defined as \(\mathbf{u} = [u_1, u_2, \ldots, u_M]^T\) and the generalized coordinates are partitioned into output coordinates \(\mathbf{u}\) and other coordinates \(\mathbf{z}\), where \(\mathbf{z}\) includes input and intermediate coordinates. The boundary to the workspace is established using a row rank deficiency criteria of the constraint Jacobian such that the boundary \(\partial A\) is defined by

\[ \partial A \subset \{ \mathbf{u} \in A : \text{Rank} \left[ \Phi(u, z) \right] < m, \]  

for some \(z\) with \(\Phi(u, z) = 0\)

where \(\Phi\) is the analytic Jacobian defined by

\[ \Phi_z(u, z) = \frac{\partial \Phi}{\partial z} . \]  

The row rank deficiency condition is reduced to analytical form by introducing a vector \(\gamma\) that is the basis to the null space of \(\Phi_z\) such that

\[ \Phi_z^T(u, z)\gamma = 0 \]  

and by requiring a unit vector value such that

\[ \gamma^T\gamma = 1. \]  

The solution in the inflated space of \(x = [u^T, z^T, \gamma^T]^T \in \mathbb{R}^n, n = m + n\) can be obtained by projecting solutions of the matrix composed of Eqs (1), (4), and (5) as

\[ G^*(x) = \begin{bmatrix} \Phi(u, z) \\ \Phi_z^T(u, z) \\ \gamma^T \gamma - 1 \end{bmatrix} = 0 \]  

where constraint equations of the form

\[ q_i^{\text{min}} \leq q_i \leq q_i^{\text{max}} \]  

are transformed into an equation by introducing a new generalized coordinates \(\lambda_i\) such that the inequality constraint can be rewritten as

\[ q_i = a_i + b_i \sin \lambda_i \]  

where \(a_i = (q_i^{\text{max}} + q_i^{\text{min}})/2\) and \(b_i = (q_i^{\text{max}} - q_i^{\text{min}})/2\) are the mid-point and half-range of the inequality constraint.\textsuperscript{10} However, in order to trace one-dimensional numerical curves, it is necessary to
introduce a cutting plane defined by the linear relations among the output coordinates \( \mathbf{u} \) of the form

\[
\mathbf{L} \mathbf{u} = \mathbf{b} \quad (9)
\]

where the matrix \( \mathbf{L} \) is of dimension \((m+2) \times m\). Therefore, augmenting Eq. (6) with Eq. (9) yields a one-dimensional set vector constraint function (\( \mathbf{b} \) is a selected value)

\[
\mathbf{G}(\mathbf{x}) = 
\begin{bmatrix}
\Phi(\mathbf{u}, \mathbf{z}) \\
\Phi^T(\mathbf{u}, \mathbf{z}) \\
\mathbf{y}^T \mathbf{y} - 1 \\
\mathbf{L} \mathbf{u} - \mathbf{b}
\end{bmatrix} = 0. 
\quad (10)
\]

A starting point on a curve can be determined by implementing a minimization approximation using the Moore-Penrose pseudo-inverse in conjunction with the Moore-Penrose pseudo-inverse where a solution to

\[
\Phi(\mathbf{u}^*, \mathbf{z}^*) \Delta \mathbf{q}^* = -\Phi(\mathbf{u}^*, \mathbf{z}^*) 
\quad (11)
\]

is sought. The Moore-Penrose inverse is defined by

\[
\Phi^* = \Phi^T(\Phi \Phi^T)^{-1} 
\quad (12)
\]

and the minimization problem is solved to obtain

\[
\Delta \mathbf{q} = \Phi^*(\mathbf{x}). 
\quad (13)
\]

Using the iterative solution method, steps of length \( h \) are taken until a boundary curve is reached.

In order to trace one-dimensional continuation curves on the boundary to the workspace, a well-established computer code called PITCON \(^{22}\) is used. A unique tangent \( h(\mathbf{x}) \) along the curve is uniquely defined by

\[
\mathbf{G}_h(\mathbf{x}) \cdot h(\mathbf{x}) = 0 
\quad (14)
\]

\[
h^T(\mathbf{x}) = 1
\quad (15)
\]

\[
\mathbf{D}_h \left[ \mathbf{G}_h(\mathbf{x}) \right] > 0. 
\quad (16)
\]

Bifurcation points along the curves are encountered for which the matrix of Eq. (16) becomes rank-deficient more than one. These bifurcation points are identified in the work by Haug et al. \(^{11}\) as higher-order singularity points. A method for identifying these points was presented by Adkins \(^2\) and validated using the spatial Stewart platform. Second-order necessary conditions for tangents are employed to form a system of second-order polynomial equations that is solved using HOMEPACK. \(^{32}\)

2.2. Method II: boundary surfaces to the workspace

Using the Denavit–Hartenberg \(^2\) representation, the position of the end-effector can be analytically formulated as

\[
\mathbf{x} = \Omega(\mathbf{q}) 
\quad (17)
\]

where \( \mathbf{q} = [q_1, q_2, \ldots, q_n] \in \mathcal{A} \) is the vector of joint coordinates. Joint limit constraints are imposed using the transformation defined in Eq. (8) as \( \mathbf{q} = \Psi(\lambda) \). For any admissible configuration, the following \((n+3)\) augmented constraint equations must be satisfied:

\[
\begin{bmatrix}
\mathbf{x} - \Omega(\mathbf{q}) \\
\mathbf{q} - \Psi(\lambda)
\end{bmatrix} = 0 
\quad (18)
\]

where the augmented vector \( \mathbf{q}^* = [\mathbf{x}^T, \mathbf{q}^T, \lambda]^T \).

Singularity sets resulting from a row-rank deficiency criterion must be determined to generate envelopes to the workspace. These envelopes are characterized by surface patches that exist inside, outside and extending both internal and external to the workspace. The sub-Jacobian \( \mathbf{H}_s \) can be evaluated as

\[
\mathbf{H}_s = 
\begin{bmatrix}
-\Omega(\mathbf{q}) & 0 \\
\mathbf{I} & \mathbf{D}
\end{bmatrix}
\quad (19)
\]

which is an \((n+3) \times (2n)\) matrix, where \( \Omega_\mathbf{q} = \partial \Omega / \partial \mathbf{q} \) is a \((3 \times n)\) matrix, \( \mathbf{I} \) is an \((n \times n)\) identity matrix, and \( \mathbf{D} \) is a diagonal matrix with the diagonal elements as

\[
\mathbf{D}_h = h_i \cos \lambda_i. 
\quad (20)
\]

The objective is to find the constant subvectors of \( \mathbf{q} \), denoted by \( \mathbf{q}_s \), which makes the sub-Jacobian \( \mathbf{H}_s \) row-rank deficient. Three singularity types are identified:

- Jacobian singularities (called type I) that satisfy

\[
S^{(1)} \equiv \mathbf{s} \in \mathbb{Q}: \text{Rank} \Omega_\mathbf{q}(\mathbf{u}, \mathbf{s}) < 3, \text{ for some constant } \mathbf{s}. 
\quad (21)
\]

- A set characterized by a null space criterion imposed on the reduced-order manipulator (called type II singular set):

\[
S^{(2)} = \left\{ \mathbf{s} \in \mathbb{Q}: \dim \left[ \text{Null} \left( \Phi_\mathbf{q}^*(\mathbf{q}^*) \right) \right] \geq 1, \text{ for some } \mathbf{q}^* \right\}. 
\quad (22)
\]

where \( \Phi_\mathbf{q}^* \) is the Jacobian after reducing the order of the manipulator (by substituting a joint limit) with respect to the reduced coordinates \( \mathbf{q}^* \).

- The set defined by a combination of all constant generalized coordinates (called type III singular set):

\[
S^{(3)} = \left\{ \mathbf{s} \in \mathbb{Q}: \left[ q_i \ q_j \right], \text{ for } i, j = 1 \text{ to } n; i \neq j \right\}. 
\quad (23)
\]

Substituting these singular sets into the position vector defined by Eq. (17) yields parametric singular geometric entities (curves or surfaces) defined by

\[
\mathbf{n}(\mathbf{u}) \equiv \Omega(\mathbf{u}, \mathbf{s}). 
\quad (24)
\]

Intersections between these singular surfaces may exist. Moreover, these curves partition a singular surface into a number of regions called subsurfaces.
denoted by \( \psi^\iota \). These curves are determined by computing the intersection curves of

\[
N(u^\iota) - N^2(u^\iota) = 0. \tag{25}
\]

It is now possible to identify surface patches that are on the boundary using a perturbation technique first introduced by Abdel-Malek and Yeh.\(^2\) At any point on the subsurface \( \psi(u, v) \), where \( u \) and \( v \) are the parameters, and for a corresponding joint coordinate vector \( q^\iota \) defined as \( q^\iota = [u^\iota, v^\iota, s^\iota]^T \), where \( s \) is the constant singular vector of the singular surface, the normal vector can be computed\(^7\) as

\[
N^\iota = \left( \frac{\partial \psi^\iota}{\partial u} \times \frac{\partial \psi^\iota}{\partial v} \right) \left/ \left| \frac{\partial \psi^\iota}{\partial u} \times \frac{\partial \psi^\iota}{\partial v} \right| \right. \tag{26}
\]

For a small perturbation \( \partial \psi \) about point \( q^\iota \) along the normal vector \( N^\iota \), the coordinates of the perturbed point can be calculated as

\[
q^\iota_{1,2} = \psi(u^\iota, v^\iota) \pm \delta u N^\iota. \tag{27}
\]

For the perturbed point to exist within the workspace, there should exist an admissible joint coordinate vector, which satisfies Eq. (17) subject to joint constraints such that a solution to the following equation exists.

\[
\begin{bmatrix}
x^\iota - \Phi(q) \\
x - \Psi(q)
\end{bmatrix} = 0 \quad j = 1, 2. \tag{28}
\]

The non-linear system of equations characterized by Eq. (28) has seven constraints with eight variables. The Moore–Penrose pseudo-inverse is used to converge to a solution. The subsurface \( \psi \) is an internal surface if and only if there exist solutions of Eq. (28) for both perturbations of \( \pm \delta u \). Otherwise, it is on the boundary of the workspace.

3. EXAMPLE

To illustrate both formulations, consider the four-degrees-of-freedom manipulator shown in Fig. 1. Joint limits are imposed as \( 0 \leq q_1 \leq 2\pi, 20 \leq q_2 \leq 50, \), \(( -\pi/4) \leq q_3 \leq \pi, \) and \( 10 \leq q_4 \leq 20 \).

For a point \( P \) on the end-effector, shown in Fig. 5, the position vector is

\[
x(q) = [q_4 \cos q_1 \cos q_3 + 30 \cos q_3 q_4 \sin q_1 \cos q_3 + 30 \sin q_1 q_4 \sin q_3 + q_2]^T.
\]

Using method I, the constraint equation is derived as

\[
\Phi(q) = \begin{bmatrix}
x - q_4 \cos q_1 \cos q_3 - 30 \cos q_3 \\
y - q_4 \sin q_1 \cos q_3 - 30 \sin q_3 \\
z - q_4 \sin q_3 - q_2
\end{bmatrix} \tag{30}
\]

where \( q = [x, y, z, q_1, q_2, q_3, q_4]^T \). The unilateral constraints defined above are enforced by replacing the generalized coordinates \( q \) with those in terms of the new coordinates \( \lambda_i \). Continuation curves computed for cutting plane \( y = 0 \) are shown in Fig. 2.

Bifurcation points encountered along curves are identified by the symbol (\( \bigcirc \)) in the figure. Non-repeated eigenvalues computed using method I correspond to a simple bifurcation point, with physical space tangent as a unit vector with one in the seventh entry and the matrix \( \Phi^T \Phi \) is rank deficient one. Bifurcation points numbered 1–4 in Fig. 2 have two tangents at which \( \Phi^T \Phi \) is row rank deficient two. Bifurcation points 5–8 are of the type for which \( \Phi^T \Phi \) is row rank deficient two, indicating a multiple of tangents. Bifurcation points 9–12 are simple bifurcation points.

Using method II, the position vector function of a point on the end-effector is given by

\[
\Omega(q) = \begin{bmatrix}
q_4 \cos q_1 \cos q_3 + 30 \cos q_3 \\
q_4 \sin q_1 \cos q_3 + 30 \sin q_3 \\
q_4 \sin q_3 - q_2
\end{bmatrix} \tag{31}
\]

the Jacobian is derived as

\[
\Omega(q) = \begin{bmatrix}
-q_4 \cos q_1 \cos q_3 - 30 \sin q_1 \\
q_4 \sin q_1 \cos q_3 + 30 \cos q_3 \\
0
\end{bmatrix} \tag{32}
\]

Inequality constraints are parametrized as \( q_1 = \pi + \pi \sin \lambda_1, \quad q_2 = 35 + 15 \sin \lambda_2, \quad q_3 = \pi + \pi \sin \lambda_3, \) and \( q_4 = 15 + 5 \sin \lambda_4. \)

There are no singular sets due to type I (Jacobian singularities) because the solutions obtained from the determinants of the sub-Jacobians do not satisfy the
joint inequality constraints. Therefore, the first singularity set is null. Type-II singularity sets are calculated by fixing one joint at its limit and solving for the remaining sub-Jacobian. There are four type-II singular sets defined by $s_1 = \{q_2 = 0, q_4 = 10\}$, $s_2 = \{q_3 = \pi, q_4 = 10\}$, $s_3 = \{q_3 = 0, q_4 = 20\}$, and $s_4 = \{q_3 = \pi, q_4 = 20\}$. Type-III singularity sets are readily obtained from the possible combinations of any two joints reaching their limits. Therefore, 10 type-III singular sets are defined by $s_5 = \{q_2 = 20, q_3 = -\pi/4\}$, $s_6 = \{q_2 = 20, q_3 = \pi\}$, $s_7 = \{q_2 = 20, q_4 = 10\}$, $s_8 = \{q_2 = 20, q_4 = 20\}$, $s_9 = \{50, q_3 = -\pi/4\}$, $s_{10} = \{q_2 = 50, q_3 = \pi\}$, $s_{11} = \{q_2 = 50, q_4 = 10\}$, $s_{12} = \{q_2 = 50, q_4 = 20\}$, $s_{13} = \{q_3 = -\pi/4, q_4 = 10\}$, and $s_{14} = \{q_3 = -\pi/4, q_4 = 20\}$. Substituting each singular set into Eq. (31) yields parametric equations of singular surfaces in $\mathbb{R}^3$. For example, substituting $s_8 = \{q_2 = 20, q_4 = 20\}$ into Eq. (31) yields the parametric equation of surface $\mathbb{R}^8$ shown in Fig. 3. There are a total of 14 such surfaces.

Performing the intersection between these surfaces defined yields singular curves that partition each surface into a number of subsurfaces. Performing the perturbation method identifies those subsurfaces that are on the boundary to the workspace. The complete workspace is shown in Fig. 4.

A cross-section of the workspace volume depicting singular surfaces is shown in Fig. 5.

Figure 6 depicts a cross-section (a longitudinal slice) due to $q_1 = \pi/2$. This cross-section is indeed identical to the results from method I. Therefore, the
continuation curves are in fact the projection of the singular surfaces on to a plane cutting through the workspace.

4. COMPARISON

Although both methods yield a representation of the boundary to the workspace, both methods have limitations. These limitations and some of the difficulties encountered by both methods are discussed in this section.

4.1. Limitations

Because method II relies upon obtaining an analytical expression for the Jacobian in order to determine all singularities, it is necessary to have a representation of the position vector on the end-effector as $x = \Omega(q)$, explicitly in terms of the generalized coordinates. Since for a closed-loop mechanism it is often impossible to write the position vector explicitly in terms of the generalized coordinates, method II cannot be used to generate boundary surfaces. In method I, closed-loop mechanisms can be studied.

4.2. Difficulties in determining all solutions

In method I, curves on the boundary are first detected using a ray emanating from an assembled configuration of the manipulator at $q^{e} = [w^{T}, x^{T}]^{T}$, with $w^{T}$ in $A$, as shown schematically in Fig. 7.

A unit vector $e$ in the output space is selected, and the ray emanating from $w^{e}$ in $A$ along vector $e$ is traced until a boundary $\partial A$ is encountered. Using this method, and because of the discretization, it is possible that the ray misses some curves inside the workspace, therefore failing to identify curves bounding enclosures (also called voids) inside the workspace.

In method II, although most of the singularities are determined from a combination of all joint limits, difficulties may be encountered in determining all singular sets from the highly non-linear equations. Currently, manual manipulation of the equations (using a symbolic manipulator called MATHEMATICA®) is used to determine those sets. Therefore, both methods have difficulties in obtaining all relevant boundaries.

4.3. Determining boundary entities

In method I, a ray emanating from a point internal to the workspace is cast and all intersections with numerical curves are detected using a stepwise approach. To illustrate, consider the continuation curves depicted in Fig. 8(a) resulting from tracing the motion of a planar Stewart platform. Figure 8(b) depicts a ray emanating from point $Q$ and intersecting with continuation curves at points marked by a square. After the numerical algorithm for detecting the boundary is completed, curves only on the boundary are depicted in Fig. 8(c).

At these intersections, numerical curves are identified whether inside or outside the boundary to the workspace. Difficulties may be encountered first in hitting every curve, and, second, when the ray passes through a bifurcation point, it may not be possible to identify which branch is internal and which is external to the boundary.

In method II, singular surfaces inside the workspace are intersected to form subsurfaces that may
exist inside or on the boundary to the workspace. Once those surface patches are determined, the perturbation method presented above is used to identify whether the subsurface (surface patch) is on the boundary. However, a difficulty may arise in partitioning a singular surface to subsurfaces due to the nature of the parametric equation of singular surfaces. To illustrate, consider the three-degrees-of-freedom manipulator shown in Fig. 9.

The position vector can be written as

\[ x = \Omega(q) = \begin{bmatrix} 10 \cos q_1 \cos q_2 \cos q_3 - 10 \sin q_1 \sin q_1 - 10 \cos q_1 \cos q_2 + 20 \cos q_1 \cos q_2 \\ 10 \sin q_1 \cos q_2 \cos q_3 + 10 \cos q_1 \sin q_1 - 10 \sin q_1 \sin q_2 + 20 \sin q_1 \cos q_2 \\ 10 \sin q_2 \cos q_3 + 10 \cos q_2 + 20 \sin q_2 + 50 \end{bmatrix} \]  

subject to the following joint limits:

\[-\frac{\pi}{2} \leq q_1 \leq \frac{\pi}{2}, -\frac{\pi}{2} \leq q_2 \leq \frac{\pi}{2}, \text{ and } 0 \leq q_3 \leq 2\pi.\]

Singular sets were determined by Abdel-Malek and Yeh.\(^2\) Substituting these singularities into Eq. (33) yields parametric equation of singular surfaces that intersect in the workspace. For example, in order to partition the singular surface \(H^3\) into subsurfaces, it is necessary to determine all intersections of \(H^3\) with all other singular surfaces. The trace of all other surfaces is shown in Fig. 10, where \(e^1\) and \(e^2\) are the curves resulting from the intersection with \(H^6\). The surface also intersects with \(H^6\) and \(H^1\) in \(e^3\) and \(e^4\), respectively. Therefore, the surface is partitioned as shown in Fig. 10.

Note that difficulties may be encountered in partitioning, since some of the curves overlap rendering it difficult to compute the intersection between the curves to identify the boundary. The perturbation technique can then be used to identify each subsurface, whether it is on the boundary or inside the workspace. For example, subsurface \(\psi^1\) is an internal subsurface while subsurface \(\psi^2\) is a boundary to the workspace.

### 4.4. Higher-order bifurcation points

In the tracing method, as the numerical algorithm is tracing a continuation curve, it is often the case that the code will identify a singularity point that has a multiple of tangents (called a higher-order bifurcation point), where only a few of the tangents can be traced. This case may arise at bifurcation points that have three tangents. Such a case is characterized by points 5-8. A branch may, therefore, be missed if care is not taken in identifying starting points on the curves and tracing until reaching these bifurcation points.
points. It is now clear that these curves are intersections of all singular surfaces, defined by method II. Therefore, continuation curves, previously unexplained in physical space, are now identified as the trace of a singular surface at which the manipulator loses at least one degree of freedom. The intersection of two singular surfaces, cut by a plane, specified by method I as a bifurcation point, is now identified in physical space as the projection of the intersection of two singular surfaces onto a plane. Since at a bifurcation point there may exist two singular surfaces, the manipulator loses at least two degrees of freedom. To illustrate, consider the intersection of singular surfaces $\Sigma^3$, $\Sigma^4$ and $\Sigma^6$ as shown in Fig. 11, for which the singularities are $s_a = (q_3 = \pi, q_4 = 20)$, $s_b = (q_2 = 20, q_3 = \pi)$ and $s_c = (q_2 = 20, q_4 = 20)$. The common constant generalized coordinates are $q_2 = 20, q_3 = \pi$ and $q_4 = 20$. Therefore, the actual intersection in space yields a curve for which only joint 1 ($q_1$) is allowed to rotate generating a curve. The projection of this curve on a cutting plane used in method I, yields the so-called bifurcation point (point 2 on Fig. 2).

4.5. Depicting surfaces patches on the boundary to the workspace

In method I, because the resulting curves are numerical in nature taken at different cross-sectional planes, it is necessary to patch these curves by connecting them with known surfaces. For example, a Bezier patch may be used to patch a region on the surface. In method II, surfaces are determined on the boundary in parametric form and are depicted using a symbolic manipulator (MATHEMATICA®). However, depicting surfaces that are parametrized in more than two parameters may be sometimes difficult to achieve.33

4.6. Impediments to motion

Using two different approaches, it is possible to overcome difficulties in controllability encountered by the end-effector when maneuvering across singular surfaces inside the workspace. In method I, barriers to motion are identified using the definite- ness properties of a quadratic form resulting from the differentiation of a normal movement functional at points along continuation curves. In method II, normal acceleration analysis with respect to curvature properties of each singular surface also yields a quadratic form whose definiteness properties determine the crossability of a subsurface. Both methods are completely different, yet yield identical results as validated by several examples3,33 demonstrated for the four-degrees-of-freedom manipulator by solid lines in Fig. 12. These lines are surfaces inside the workspace upon which the manipulator may exhibit control difficulties.

5. APPLICATIONS

Several applications pertaining to both methods have been identified some of which have been implemented. Method I has been used to determine domains of interference for complex mechanisms, operational envelopes of rigid bodies moving in space, and the dextrous workspace of mechanical manipulators. Method II has been used in the determination of

Fig. 11. Intersection of singular surfaces $\Sigma^3$, $\Sigma^4$ and $\Sigma^6$. 
the swept volume of a solid sweeping along a geometric entity (with multiple parameters), used in the NC verification of manufactured parts, and in determining a measure of dexterity for serial robot manipulators.

6. CONCLUSIONS

The accumulation of two independent works dealing with the determination of manipulator workspaces and their application have been presented. The compilation of several works dealing with the use of continuation methods to trace numerically curves that are boundary to the workspace. Several works are also compiled presenting recent developments in the determination of parametric surface patches bounding the workspace.

It was shown that continuation curves resulting from method I are indeed the projection on to a plane of singular surfaces used in method II. It was also shown that, although both methods yield identical results, both methods are not yet mature enough for completely automatic implementation into computer code since both methods require engineering insight into the functionality of the mechanism. Nevertheless, both methods have demonstrated feasibility in determining the boundary to the workspace.

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