Chapter 11
3D Vision, Geometry

**Topics:**
- Basics of projective geometry
  - Points and hyperplanes in projective space
  - Homography
  - Estimating homography from point correspondence
- The single perspective camera
  - An overview of single camera calibration
  - Calibration of one camera from the known scene
- Scene reconstruction from multiple views
  - Triangulation
  - Projective reconstruction
  - Matching constraints
  - Bundle adjustment
- Two cameras, stereopsis
  - The geometry of two cameras. The fundamental matrix
  - Relative motion of the camera; the essential matrix
  - Estimation of a fundamental matrix from image point correspondences
  - Camera Image rectification
  - Applications of the epipolar geometry in vision
- Three and more cameras
  - Stereo correspondence algorithms
**Epipolar geometry and Fundamental matrix**

Fundamental matrix relates corresponding points in two stereo images

\[ u'^T F u = 0 \]

What does it mean?
A point on the left image ≈ a line on the right image
What is this line called
Fundamental matrices relating multiple cameras
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Fundamental matrices relating multiple cameras
Image rectification (before)
Image rectification (after)
**Image rectification**

What happens in terms of epipolar geometry?

Where are the two epipoles?

What is the relation between the baseline and the camera matrix?

Can we solve it using a homographic transformation on each camera image?
**Image rectification**

What happens in terms of epipolar geometry?

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Can we solve it using a homographic transformation on each camera image?
Image rectification: advantages

3D reconstruction becomes easier

Image stitching to generate a panoramic view
Panoramic view
So, how to accomplish image rectification?

- Learn how to determine the fundamental matrix
- Relative camera motion and essential matrix
- Relation between fundamental matrix and camera matrix
- Compute image rectification
Relative camera motion and essential matrix

In the previous class, we have seen:  \( F = K'^{-T}RS(t)K^{-1} \)

\( K' \) and \( K \) are intrinsic camera parameters that maps Euclidean image plane to image pixels; primarily plays a role to correct the shear distortion between the \( x \)- and \( y \)-axes.

It’s very difficult to determine \( K' \) and \( K \) without use of a known 3D scene and just by using the correspondence between two acquired images

Thus, if we ignore this shear component, the epipolar constraint in the image Euclidean plane translates to

\[ u'_iTRS(t)u_i = 0 \Rightarrow u'_iT E u_i = 0, \quad \text{where } E = RS(t) \]

\( E \) is called the **essential matrix** that defines the relative motion between two camera position


Relation between fundamental matrix and essential matrix (when we know \( K' \) and \( K \) )

\[ E = K'^TFK \]
Decomposition of essential matrix

Note that the vector $\mathbf{t}$ in the essential matrix $E = RS(\mathbf{t})$ tells us about the relative location of the two optical centers. i.e., the baseline.

Also, assuming that the camera matrix $M = [I \mid \mathbf{0}]$ for the first camera, $R$ and $\mathbf{t}$ together determine $M'$ -- the camera matrix of the second camera.

Now, assume that, somehow, we have computed the essential matrix $E$. But, it does not immediately give us the translation vector $\mathbf{t}$ or the rotation matrix $R$.

So, we need to decompose $E$.

Singular value decomposition of $E$ gives $E = U\Sigma V^T$, $U$ and $V$ are rotation matrices.

Following that the rows of $S(\mathbf{t})$ are coplanar (why), it has a rank of two and the two singular values are equal (follows from the formulation of $S(\mathbf{t})$); so

$$D = diag[\sigma, \sigma, 0]$$

We will later see that scale factor in the actual computation of $E$ is arbitrarily set.
Decomposition of the essential matrix  

Denote 

\[
\bar{t} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \bar{R} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

Then the translation vector is given by 

\[
S(t) = VS(\bar{t})V^T
\]

The rotation matrix is not given uniquely, we have 

\[
R = U\bar{R}V^T \quad \text{or} \quad R = U\bar{R}^TV^T
\]
Before getting into image rectification, we need to learn

- Relation between the fundamental matrix and the camera matrix
- How to compute the fundamental matrix

Camera matrices:

\[ M = [I \mid 0] \]

\[ M' = [S(e')F \mid e'] \]
Computation of the fundamental matrix using point correspondence

Number of unknowns:
9 parameters in $F$ minus one for scale standardization minus one for rank of $F$ is two

$$9 - 1 - 1 = 7$$

So, we can solve $F$ with $m \geq 8$ corresponding point pairs in two images.

We have to solve the following linear system:

$$u_i' F u_i = 0, \quad i = 1, 2, \ldots, m$$

Use Kronecker product identity: $ABc = (c^T \otimes A)b$

$$u_i'^T F u_i = [u_i^T \otimes u_i'^T]f = 0$$

Put together all point correspondences

$$
\begin{bmatrix}
    u_{i,1}^T \otimes u_{i,1}'^T \\
    \vdots \\
    u_{i,m}^T \otimes u_{i,m}'^T
\end{bmatrix}f = Wf = 0
$$

Compute $W^T W$ and apply singular value decomposition; choose $f$ along the eigenvector corresponding to the smallest eigenvalue
Computation of the fundamental matrix using maximum likelihood estimation

\[
\min_{F, u_i, v_i, u'_i, v'_i} [(u_i - \hat{u}_i)^2 + (v_i - \hat{v}_i)^2 + (u'_i - \hat{u}'_i)^2 + (v'_i - \hat{v}'_i)^2]
\]

Given \([u'_i, v'_i, 1]F[u_i, v_i, 1]^T = 0\) and \(\det F = 0\)

Use Lagrange multiplier

\[
\text{maximize } f(x, y), \text{ given } g(x, y) = c
\]

is equivalent to optimizing the Lagrange function

\[
\Lambda(x, y, \lambda) = f(x, y) + \lambda \cdot (g(x, y) - c)
\]

where \(\lambda\) is the new variable called Lagrange multiplier