Chapter 7.2 Laminar Boundary Layers

Boundary Layer Theory

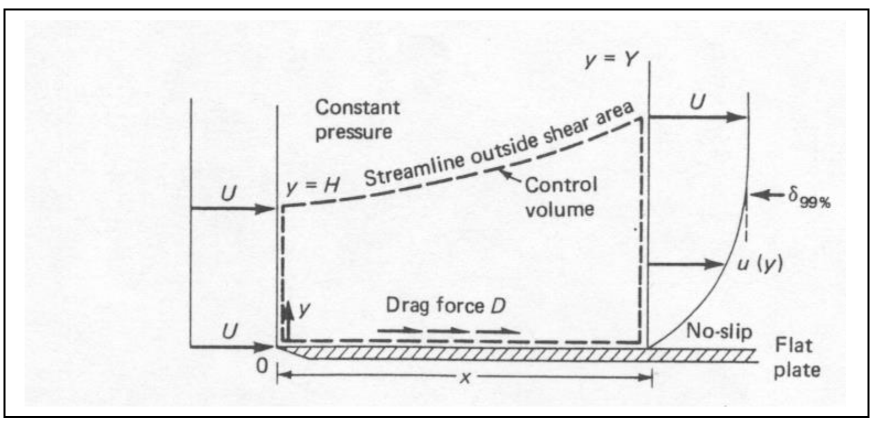
Introduction:

Boundary layer flows: External flows around streamlined bodies at high Re have viscous (shear and no-slip) effects confined close to the body surfaces and its wake but are nearly inviscid far from the body.

Applications of BL theory: *aerodynamics* (airplanes, rockets, projectiles), *hydrodynamics* (ships, submarines, torpedoes), *transportation* (automobiles, trucks, cycles), *wind engineering* (buildings, bridges, water towers), and *ocean engineering* (buoys, breakwaters, cables).

Flat-Plate Momentum Integral Analysis & Laminar approximate solution

Consider flow of a viscous fluid at high Re past a flat plate, i.e., flat plate fixed in a uniform stream of velocity $U\hat{i}$.



Boundary-layer thickness arbitrarily defined by y = $\delta_{99\%}$ (where, $\delta_{99\%}$ is the value of y at u = 0.99U). Streamlines outside $\delta_{99\%}$ will deflect an amount δ^* (**the displacement thickness**). Thus, the streamlines move outward from $y = H$ at $x = 0$ to $y = Y = \delta = H + \delta^*$ at $x = x_1$.

<u>Conservation of mass</u>:

$$\int_{CS} \rho \underline{V} \bullet \underline{n} dA = \mathbf{0} = -\int_0^H \rho U dy + \int_0^{H+\delta^*} \rho u dy$$

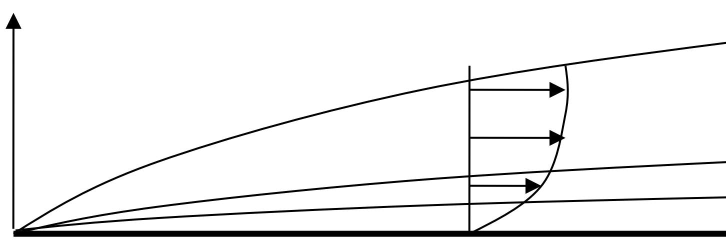
Assuming incompressible flow (constant density), this relation simplifies to

$$UH = \int_0^Y u dy = \int_0^Y (U + u - U) dy = UY + \int_0^Y (u - U) dy$$

Note: $Y = H + \delta^*$, we get the definition of displacement thickness:

$$\delta^* = \int_0^Y \left(1 - \frac{u}{U}\right) dy$$

δ^* (a function of x only) is an important measure of effect of BL on external flow. To see this more clearly, consider an alternate derivation based on an equivalent discharge/flow rate argument:



$$\underbrace{\int_{\delta^*}^{\delta} U dy}_{\text{Inviscid flow about } \delta^* \text{ body}} = \int_0^{\delta} u dy$$

Flowrate between δ^* and δ of inviscid flow=actual flowrate, i.e., inviscid flow rate about displacement body = equivalent viscous flow rate about actual body

$$\int_0^{\delta} U dy - \int_0^{\delta^*} U dy = \int_0^{\delta} u dy \Rightarrow \delta^* = \int_0^{\delta} \left(1 - \frac{u}{U}\right) dy$$

w/o BL - displacement effect=actual discharge

For 3D flow, in addition it must also be explicitly required that δ^* is a stream surface of the inviscid flow continued from outside of the BL.

<u>Conservation of x-momentum</u>:

$$\sum F_x = -D = \int_{CS} \rho u \underline{V} \bullet \underline{n} dA = -\int_0^H \rho U\left(U dy\right) + \int_0^Y \rho u\left(u dy\right)$$

$Drag = D = \rho U^2 H - \int_0^Y \rho u^2 dy$ = Fluid force on plate = - Plate force on CV (fluid)

Again, assuming constant density and using continuity: $H = \int_0^Y \frac{u}{U} dy$

$$D = \rho U^2 \int_0^Y u/U dy - \int_0^Y u^2 dy = \int_0^x \tau_w dx$$

$$\frac{D}{\rho U^2} = \theta = \int_0^Y \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

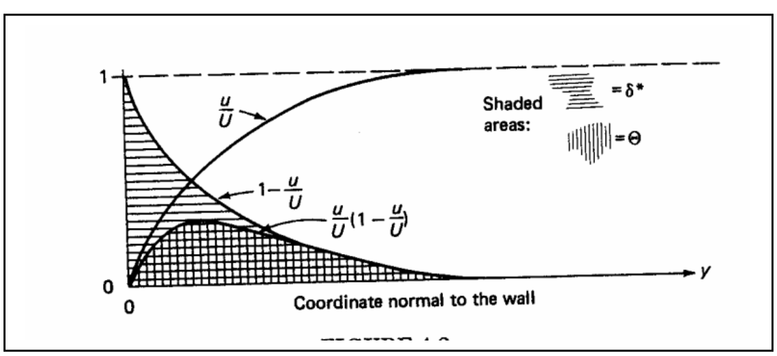
where, θ is the **momentum thickness (a function of x only)**, an important measure of the drag.

$$C_D = \frac{2D}{\rho U^2 x} = \frac{2\theta}{x} = \frac{1}{x}\int_0^x C_f dx \quad \text{Per unit span}$$

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} \Rightarrow C_f = \frac{d}{dx}\left(x C_D\right) = 2\frac{d\theta}{dx}$$

$$\frac{d\theta}{dx} = \frac{C_f}{2} \qquad \tau_w = \rho U^2 \frac{d\theta}{dx}$$

Special case 2D momentum integral equation for $p_x = 0$



Simple velocity profile approximations:

$$u = U(2y/\delta - y^2/\delta^2)$$

u(0) = 0 no slip
$\left.\begin{array}{l} u(\delta) = U \\ u_y(\delta)=0 \end{array}\right\}$ matching with outer flow

Use velocity profile to get $C_f(\delta)$ and $\theta(\delta)$ and then integrate momentum integral equation to get $\delta(Re_x)$

$\delta^* = \delta/3$
$\theta = 2\delta/15$
$H= \delta^*/\theta= 5/2$

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \mu 2U/\delta$$

$$\Rightarrow C_f = \frac{2\mu U/\delta}{1/2\rho U^2} = 2\frac{d\theta}{dx} = 2\frac{d}{dx}(2\delta/15)$$

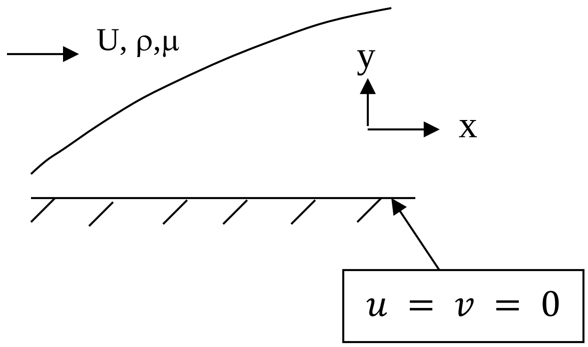
$$\therefore \delta d\delta = \frac{15\mu dx}{\rho U}$$

$$\left.\begin{array}{l} \delta^2 = \dfrac{30\mu dx}{\rho U} \\ \delta/x = 5.5/\mathrm{Re}_x^{1/2} \\ \mathrm{Re}_x = Ux/\nu \\ \delta^*/x = 1.83/\mathrm{Re}_x^{1/2} \end{array}\right\} \quad \text{10\% error, cf. Blasius}$$

$$\theta/x = 0.73/\mathrm{Re}_x^{1/2}$$

$$C_D = 1.46/\mathrm{Re}_L^{1/2} = 2C_f(L)$$

Boundary layer approximations, equations, and comments



2D NS, ρ=constant, neglect g

$$u_x + v_y = 0$$

$$u_t + uu_x + vu_y = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu(u_{xx} + u_{yy})$$

$$v_t + uv_x + vv_y = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu(v_{xx} + v_{yy})$$

Introduce non-dimensional variables that includes scales such that all variables are of O(1):

$$x^* = x/L$$

$$y^* = \frac{y}{L}\sqrt{\mathrm{Re}}$$

$$t^* = tU/L$$

$$u^* = u/U$$

$$v^* = \frac{v}{U}\sqrt{Re}$$

$$p^* = \frac{p - p_0}{\rho U^2}$$

$$Re_x = UL/\nu$$

The NS equations become (drop *)

$$u_x + v_y = 0$$

$$u_t + uu_x + vu_y = -p_x + \underline{\frac{1}{Re}u_{xx}} + u_{yy}$$

$$\frac{1}{Re}\left(v_t + uv_x + vv_y\right) = -p_y + \underline{\frac{1}{Re^2}v_{xx}} + \underline{\frac{1}{Re}v_{yy}}$$

For large Re (BL assumptions) the underlined terms drop out and the BL equations are obtained.

Therefore, y-momentum equation reduces to

$$p_y = 0$$

$$i.e.\ p = p(x,t)$$

$$\Rightarrow p_x = -\rho(U_t + UU_x)$$ From Euler/Bernoulli equation for external flow

<u>2D BL equations:</u>

$$u_x + v_y = 0$$

$$u_t + uu_x + vu_y = (U_t + UU_x) + \nu u_{yy}$$

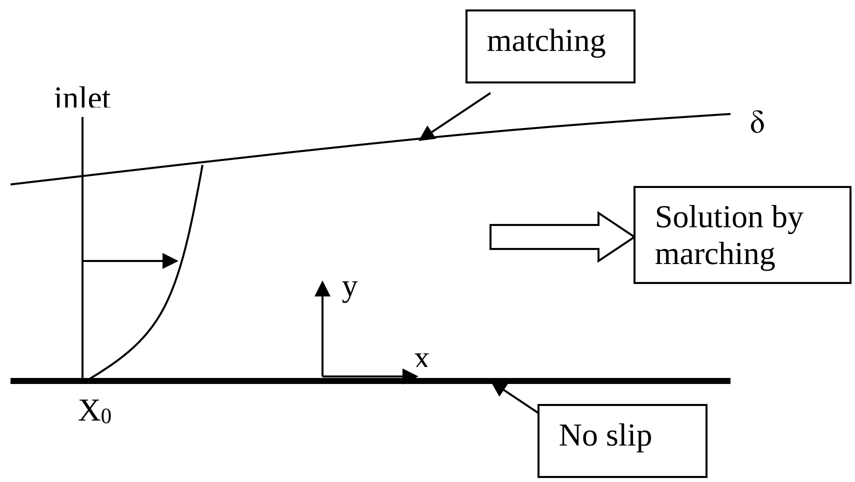
Note:

(1) U(x,t), p(x,t) impressed on BL by the external flow.

(2) $\frac{\partial^2}{\partial x^2} = 0$: i.e., longitudinal (or stream-wise) diffusion is neglected.

(3) Due to (2), the equations are parabolic in x. Physically, this means all downstream influences are lost other than that contained in external flow. A marching solution is possible.

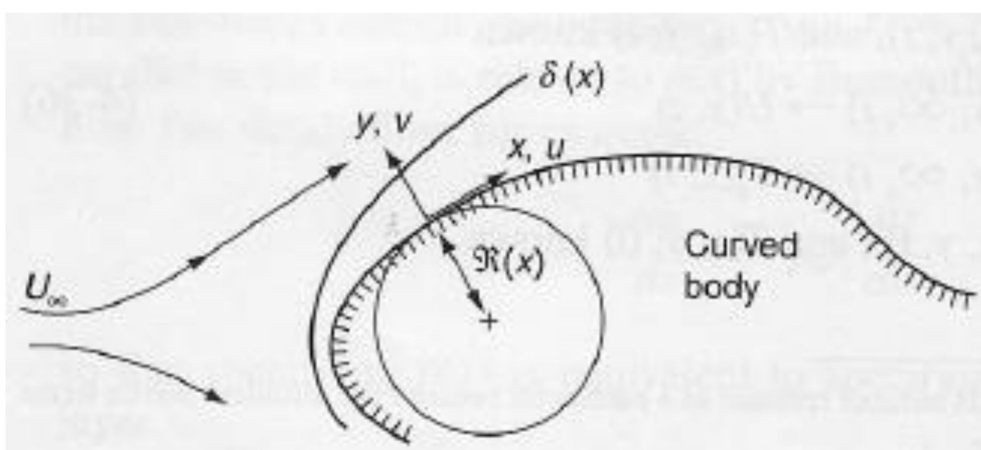
(4) Boundary conditions



No slip: $u(x,0,t) = v(x,0,t) = 0$
Initial condition: $u(x,y,0)$ known
Inlet condition: $u(x_0,y,t)$ given at x_0
Matching with outer flow: $u(x,\infty,t) = U(x,t)$

(5) When applying the boundary layer equations one must keep in mind the restrictions imposed on them due to the basic BL assumptions
→ not applicable for thick BL or separated flows (although they can be used to estimate occurrence of separation).

(6) Curvilinear coordinates



Although BL equations have been written in Cartesian Coordinates, they apply to curved surfaces provided δ << R and x, y are curvilinear coordinates measured along and normal to the surface, respectively. In such a system we would find under the BL assumptions

$$p_y = \frac{\rho u^2}{R}$$

Assume u is a linear function of y: $u = Uy/\delta$

$$\frac{dp}{dy} = \frac{\rho U^2 y^2}{R\delta^2}$$

$$p(\delta) - p(0) \propto \frac{\rho U^2 \delta}{3R}$$

Or

$\dfrac{\Delta p}{\rho U^2} \propto \dfrac{\delta}{3R}$; therefore, we require $\delta << R$

(7) Practical use of the BL theory

For a given body geometry:

(a) Inviscid theory gives p(x) → integration gives L and D = 0

(b) BL theory gives → $\delta^*(x)$, $\tau_w(x)$, $\theta(x)$, etc. and predicts separation if any

(c) If separation present then no further information → must use inviscid models, BL equation in inverse mode, or NS equation.

(d) If separation is absent, integration of $\tau_w(x)$ → frictional resistance and body + δ^*, inviscid theory gives → p(x) for body + δ^*, can go back to (b) for more accurate BL calculation including viscous – inviscid interaction

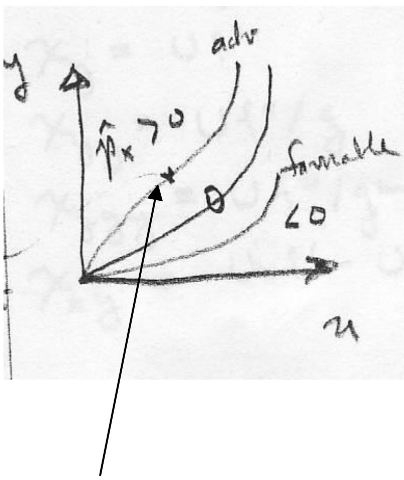
(8) Separation and shear stress

At the wall, $u = v = 0 \rightarrow u_{yy} = \frac{1}{\mu} p_x$

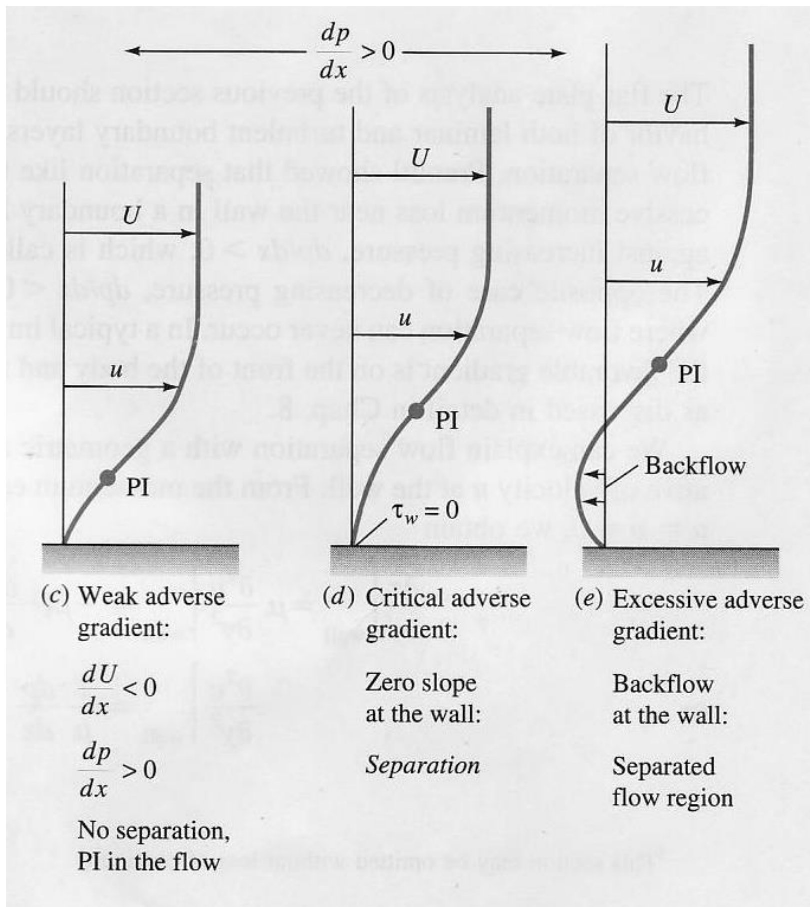
1st derivative u gives $\tau_w \rightarrow \tau_w = \mu u_y \big|_w$

$\tau_w = 0$ separation

2nd derivative u depends on p_x



Inflection point



Laminar Boundary Layer ($\boldsymbol{Re_{trans} = 5\times10^5 - 3\times10^6}$) -
Similarity solutions (2D, steady, incompressible): method of reducing PDE to ODE by appropriate similarity transformation; also, as a result of transformation at least one coordinate lacks origin such that the solution collapses to same form at all length or time scales

$$u_x + v_y = 0$$
$$uu_x + vu_y = UU_x + \nu u_{yy}$$

BCs: $u(x,0) = v(x,0) = 0$

$u(x,\infty) = U(x)$

+ inlet condition

For Similarity $\dfrac{u(x,y)}{U(x)} = F\left(\dfrac{y}{g(x)}\right)$ expect $g(x)$ related to $\delta(x)$

Or in terms of stream function ψ: $u = \psi_y \quad v = -\psi_x$

For similarity $\psi = U(x)g(x)f(\eta) \qquad \eta = y/g(x)$

$u = \psi_y = Uf' \quad v = -\psi_x = -(U_x gf + Ug_x f - Ug_x \eta f')$

BC:

$$u(x,0) = 0 \Rightarrow U(x)f'(0) = 0 \Rightarrow f'(0) = 0$$
$$v(x,0) = 0 \Rightarrow U_x(x)g(x)f(0) + U(x)g_x(x)f(0)$$
$$-U(x)g_x(x)\times 0\times f'(0) = 0$$
$$\Rightarrow \left(U_x(x)g(x) + U(x)g_x(x)\right)f(0) = 0$$
$$\Rightarrow f(0) = 0$$
$$u(x,\infty) = U(x) \Rightarrow U(x)f'(\infty) = U(x) \Rightarrow f'(\infty) = 1$$

Write boundary layer equations in terms of ψ

$$\psi_y\psi_{yx} - \psi_x\psi_{yy} = UU_x + \nu\psi_{yyy}$$

Substitute

$$\psi_{yy} = Uf''/g$$

$$\psi_{yyy} = Uf'''/g^2$$

$$\psi_{xy} = U_x f' - Uf''\eta g_x / g$$

Assemble them together:

$$(Uf')\left(U_x f' - \cancel{Uf''\eta\frac{g_x}{g}}\right) - \left(U_x g f + U g_x f - \cancel{U g_x \eta f'}\right)(Uf''/g)$$

$$= UU_x + \nu\left(U\ f'''/g^2\right)$$

$$UU_x f'^2 - UU_x ff'' - U^2\, g_x/g\ ff'' = UU_x + \nu\frac{U}{g^2}f'''$$

$$UU_x f'^2 - \frac{U}{g}(Ug)_x\, ff'' = UU_x + \nu\frac{U}{g^2}f'''$$

$$f''' + \underbrace{\boxed{\frac{g}{\nu}(Ug)_x}}_{C_1} ff'' + \underbrace{\boxed{\frac{g^2}{\nu}U_x}}_{C_2}\left(1 - f'^2\right) = 0$$

Where for similarity C_1 and C_2 are constant or function η only

- i.e., for a chosen pair of C_1 and C_2→ $U(x)$, $g(x)$ can be found (Potential flow is NOT known a priori)
- Then solution of $f''' + C_1 ff'' + C_2\left(1 - f'^2\right) = 0$ gives $f(\eta)$→ $u(x,y)$, $\tau_w = \mu\left.\frac{\partial u}{\partial y}\right|_w = \frac{\mu U f''(0)}{g}$, δ, δ*,θ, H, C_f, C_D

The Blasius Solution for Flat-Plate Flow

U=constant→ $U_x = 0 \rightarrow C_2 = 0$

Then $C_1 = \frac{U}{\nu} g g_x$

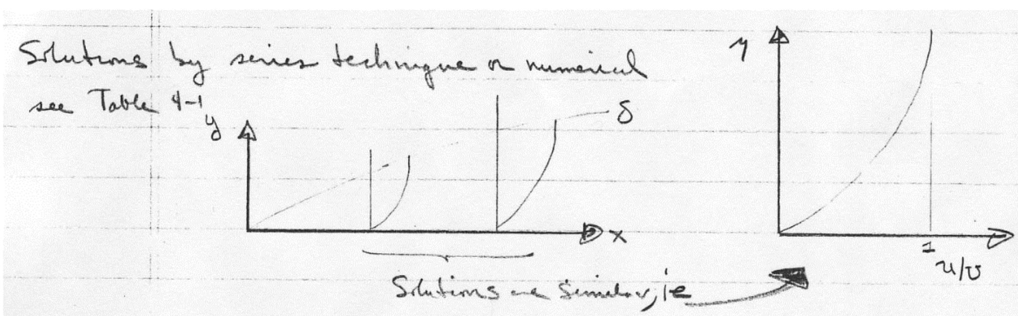
$$\frac{d}{dx}\left(g^2\right) = \frac{2C_1\nu}{U} \Rightarrow g(x) = \left[2C_1\nu x/U\right]^{1/2}$$

Let $C_1 = 1$, then $g(x) = \sqrt{\frac{2\nu x}{U}} \Rightarrow \eta = y\sqrt{\frac{U}{2\nu x}}$

$$\Rightarrow \boxed{\begin{gathered} f''' + 1/2\, f f'' = 0 \\ f(0) = f'(0) = 0,\ f'(\infty) = 1 \end{gathered}}$$

Blasius equations for Flat Plate Boundary Layer

Solutions by series technique or numerical



$\frac{u}{U} = 0.99$ when $\eta = 3.5$ → $\frac{\delta}{x} = \frac{5}{\sqrt{\mathrm{Re}_x}}$ $\mathrm{Re}_x = \frac{Ux}{\nu}$

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy = \int_0^\infty \left(1 - f'\right) d\eta \sqrt{\frac{2\nu x}{U}} \rightarrow \boxed{\frac{\delta^*}{x} = \frac{1.7208}{\sqrt{\mathrm{Re}_x}}}$$

$$\theta = \int_0^\infty \left(1 - \frac{u}{U}\right)\frac{u}{U} dy = \int_0^\infty \left(1 - f'\right) f' \sqrt{\frac{2\nu x}{U}} d\eta \rightarrow \boxed{\frac{\theta}{x} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}}$$

So, $\boxed{\dfrac{\delta^*}{\theta} = H = 2.59}$

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_w = \frac{\mu U f''(0)}{\sqrt{2\nu x/U}} \quad \rightarrow \quad \boxed{C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.664}{\sqrt{\mathrm{Re}_x}} = \frac{\theta}{x}}$$

$$\boxed{C_D = \frac{D}{\frac{1}{2}\rho U^2 L} = \int_0^L C_f \frac{dx}{L} = \frac{1.328}{\sqrt{\mathrm{Re}_L}}}; \qquad \mathrm{Re}_L = \frac{UL}{\nu};$$

$$\boxed{\frac{v}{U} = \frac{\eta f' - f}{\sqrt{2Re_x}} \ll 1} \qquad \text{for} \quad \mathrm{Re}_x >> 1$$

TABLE 4-1
Numerical solution of the Blasius flat-plate relation, Eq. (4-45)

η	$f(\eta)$	$f'(\eta)$	$f''(\eta)$
0.0	0.0	0.0	0.46960
0.1	0.00235	0.04696	0.46956
0.2	0.00939	0.09391	0.46931
0.3	0.02113	0.14081	0.46861
0.4	0.03755	0.18761	0.46725
0.5	0.05864	0.23423	0.46503
0.6	0.08439	0.28058	0.46173
0.7	0.11474	0.32653	0.45718
0.8	0.14967	0.37196	0.45119
0.9	0.18911	0.41672	0.44363
1.0	0.23299	0.46063	0.43438
1.1	0.28121	0.50354	0.42337
1.2	0.33366	0.54525	0.41057
1.3	0.39021	0.58559	0.39598
1.4	0.45072	0.62439	0.37969
1.5	0.51503	0.66147	0.36180
1.6	0.58296	0.69670	0.34249
1.7	0.65430	0.72993	0.32195
1.8	0.72887	0.76106	0.30045
1.9	0.80644	0.79000	0.27825
2.0	0.88680	0.81669	0.25567
2.2	1.05495	0.86330	0.21058
2.4	1.23153	0.90107	0.16756
2.6	1.41482	0.93060	0.12861
2.8	1.60328	0.95288	0.09511
3.0	1.79557	0.96905	0.06771
3.2	1.99058	0.98037	0.04637
3.4	2.18747	0.98797	0.03054
3.6	2.38559	0.99289	0.01933
3.8	2.58450	0.99594	0.01176
4.0	2.78388	0.99777	0.00687
4.2	2.98355	0.99882	0.00386
4.4	3.18338	0.99940	0.00208
4.6	3.38329	0.99970	0.00108
4.8	3.58325	0.99986	0.00054

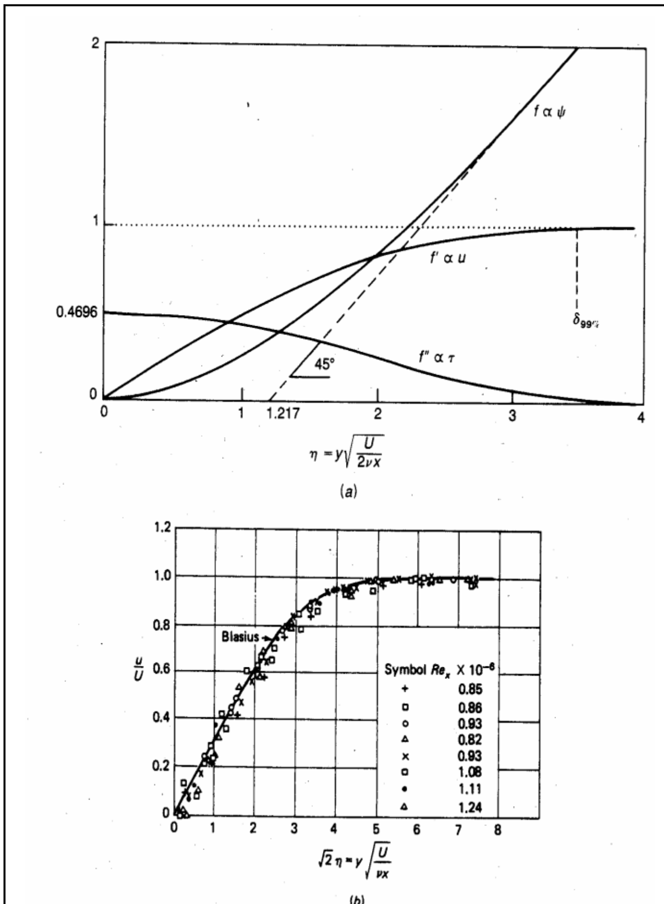
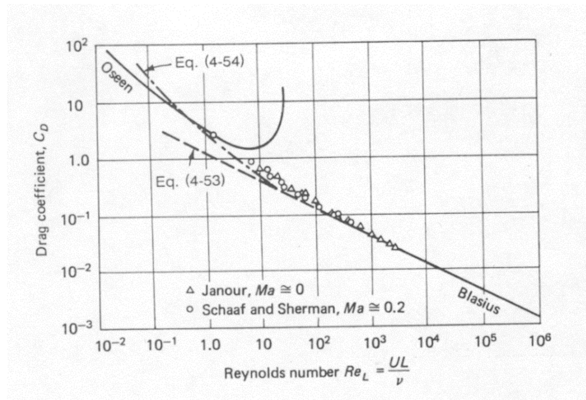


FIGURE 4-6
The Blasius solution for the flat-plate boundary layer: (*a*) numerical solution of Eq. (4-45); (*b*) comparison of $f' = u/U$ with experiments by Liepmann (1943).

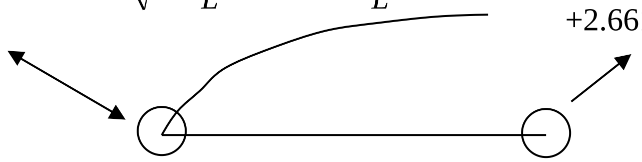
	C_D	Re_L
Oseen	3-226 (3rd edition,vicous flows)	<1
Blasius		$100<Re<Re_{tr}\sim 3\times 10^6$



LE Higher order correction

$$C_D = 1.328/\sqrt{\mathrm{Re}_L} + 2.3/\mathrm{Re}_L$$

Similar breakdown occurs at Trailing edge. From triple – deck theory the correction is $+2.661/\mathrm{Re}_L^{7/8}$



Rex small therefore local breakdown of BL approximation

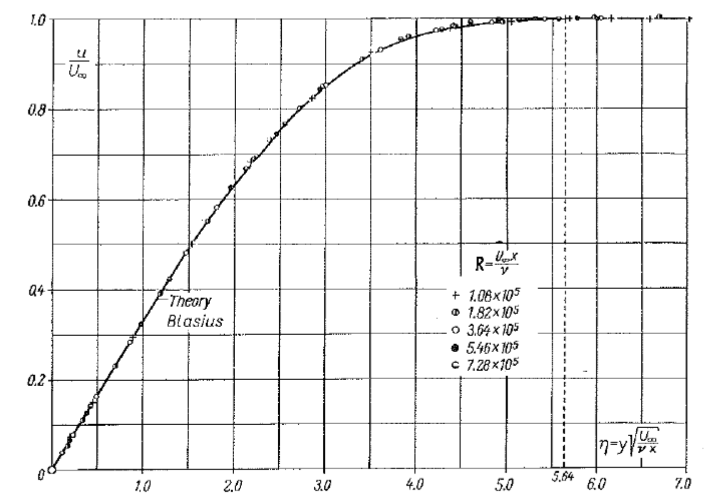


Fig. 7.9. Velocity distribution in the laminar boundary layer on a flat plate at zero incidence, as measured by Nikuradse [20]

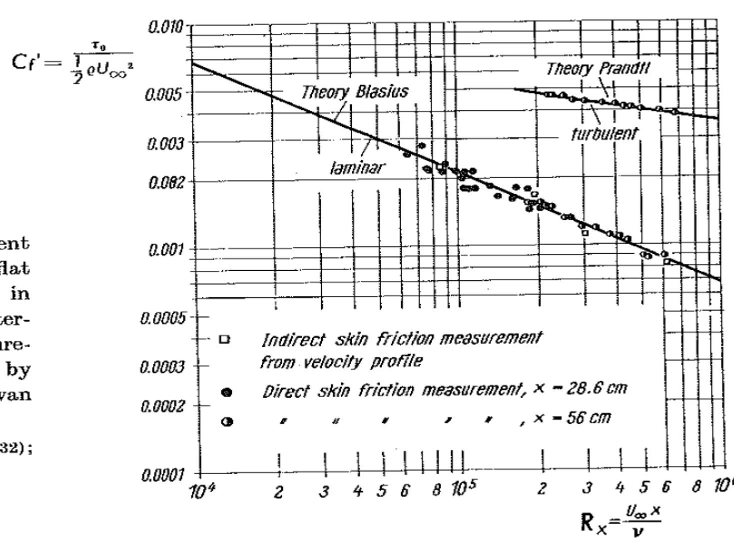


Fig. 7.10. Local coefficient of skin friction on a flat plate at zero incidence in incompressible flow, determined from direct measurement of shearing stress by Liepmann and Dhawan [6, 18]

Theory: laminar from eqn. (7.32); turbulent from eqn. (21.12)

Falkner-Skan Wedge Flows

$$f''' + C_1 f f'' + C_2\left(1 - f'^2\right) = 0$$

$$f(0) = f'(0) = 0, \qquad f'(\infty) = 1$$

$$C_1 = \frac{g}{\nu}(Ug)_x \qquad C_2 = \frac{g^2}{\nu}U_x$$

$$\left.\begin{aligned} f &= f(\eta) \\ \eta &= y/g(x) \\ u/U &= f'(\eta) \end{aligned}\right\}$$ Similarity form of BL eq.

(Blasius Solution: $C_2=0$, $C_1=1$)

Consider $$\left(Ug^2\right)_x = 2Ugg_x + g^2U_x$$
$$= 2Ugg_x + 2g^2U_x - g^2U_x$$
$$= 2g(Ug)_x - g^2U_x$$
$$= 2\nu C_1 - \nu C_2$$

Hence → $\left(Ug^2\right)_x = \nu\left(2C_1 - C_2\right)$, $\qquad C_2 = \dfrac{g^2}{\nu}U_x$

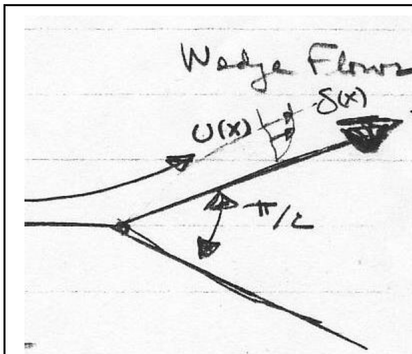
Choose $C_1=1$ and C_2 arbitrary=C,

Integrate $$Ug^2 = \nu(2-C)x$$

Combine $$\frac{U_x}{U} = \frac{C}{2-C}\frac{1}{x}$$

$$C = g^2U_x/\nu$$

$$\ln U = \frac{C}{2-C}\ln x + k$$



Then $$\left\{\begin{aligned} U(x) &= kx^{C/(2-C)} \\ g(x) &= \sqrt{\frac{\nu(2-C)}{k}}\,x^{\frac{1-C}{2-C}} \end{aligned}\right.$$

Change constants

$$U(x) = kx^m$$

$$\eta = \frac{y}{g} = y\sqrt{\frac{m+1}{2}\frac{U}{\nu x}}$$

$$f''' + ff'' + \beta\left(1 - f'^2\right) = 0, \quad \beta = \frac{2m}{m+1}, \quad m = \frac{\beta}{2-\beta}$$

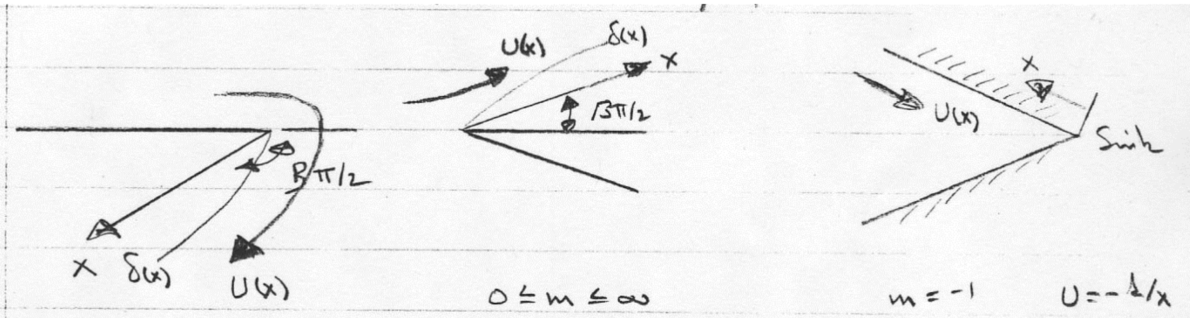
$$f(0) = f'(0) = 0 \qquad f'(\infty) = 1$$

Solutions for $-0.19884 \le \beta \le 1.0$

Separation ($\tau_w = 0$)

Solutions show many commonly observed characteristics of BL flow:

- The parameter β is a measure of the pressure gradient, dp/dx. For $\beta > 0$, $dp/dx < 0$ and the pressure gradient is favorable. For $\beta < 0$, the $dp/dx > 0$ and the pressure gradient is adverse.
- Negative β solutions drop away from Blasius profiles as separation approached
- Positive β solutions squeeze closer to wall due to flow acceleration
- Accelerated flow: $\tau_{\max}$ near wall
- Decelerated flow: $\tau_{\max}$ moves toward $\delta/2$



$-\tfrac{1}{2} \le m \le 0$

$-2 \le \beta \le 0$

Expansion Corner

$\beta = -.19884$ (Separation)

$0 \le m \le \infty$

$0 \le \beta \le 2$

Wedge Flow

$\left.\begin{matrix} m=0 \\ \beta=0 \end{matrix}\right\}$ Flat Plate

$\left.\begin{matrix} m=1 \\ \beta=1 \end{matrix}\right\}$ Stagnation Point

$m = -1$ $U = -A/x$

$\beta = \infty$ $\eta = \frac{y}{x}\sqrt{\frac{U}{\nu}}$

Sink: Simple Sol involving tanh η

Sec. 4.37

BL Sol of Jeffery-Hamel flow

Also, $\beta = 4$, $m = -2$: doublet flow near a plane wall

$\beta = 5$, $m = -5/3$: doublet near a 90° corner

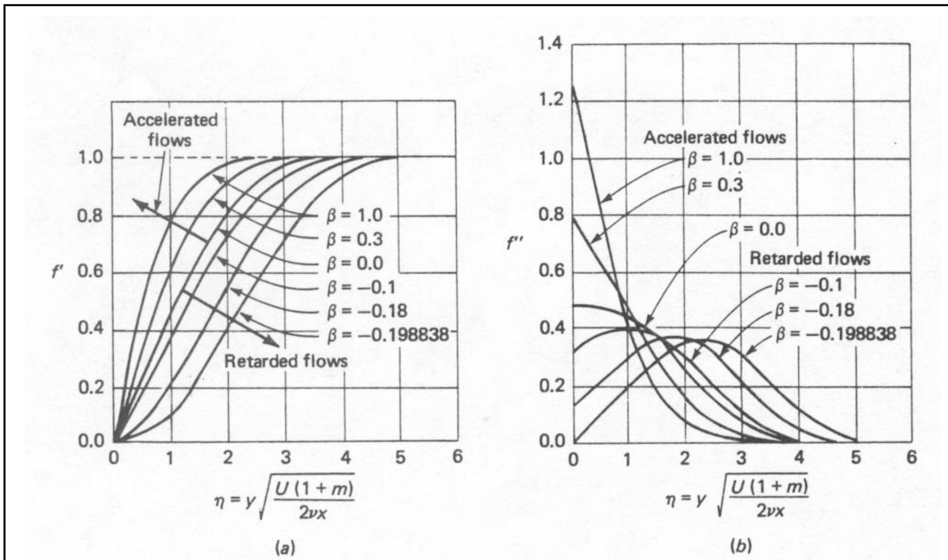


FIGURE 4-11
(a) Velocity profiles and (b) shear-stress profiles for the Falkner-Skan equation.

Momentum Integral Equation

Historically similarity and other AFD methods used for idealized flows and momentum integral methods for practical applications, including pressure gradients.

Momentum integral equation, which is valid for both laminar and turbulent flow:

$$\int_{y=0}^{\infty}\left(\text{BL form of momentum equation} + (u-U)\text{continuity}\right)dy$$

$$\frac{\tau_w}{\rho U^2} = \underbrace{\frac{1}{2}C_f = \frac{d\theta}{dx}}_{\text{For flat plate equation} \rightarrow \frac{dU}{dx}=0} + (2+H)\frac{\theta}{U}\frac{dU}{dx}$$

$$\theta = \int_0^{\delta}\frac{u}{U}\left(1-\frac{u}{U}\right)dy;$$

$$H = \frac{\delta^*}{\theta};$$

$$\delta^* = \int_0^{\delta}\left(1-\frac{u}{U}\right)dy$$

Momentum: $uu_x + vu_y = -\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right) + \frac{1}{\rho}\frac{\partial\tau}{\partial y}$

The pressure gradient is evaluated form the outer potential flow using Bernoulli equation

$$p + \frac{1}{2}\rho U^2 = \text{constant}$$

$$p_x + \frac{1}{2}\rho 2UU_x = 0$$

$$-p_x = \rho UU_x$$

$$(u-U)\underbrace{\left(u_x+v_y\right)}_{Continuity} = uu_x + uv_y - Uu_x - Uv_y$$

$$\underbrace{uu_x + vu_y - UU_x - \frac{1}{\rho}\tau_y}_{0} + \underbrace{uu_x + uv_y - Uu_x + Uv_y}_{0} = 0$$

$$-\frac{1}{\rho}\tau_y = -2uu_x - vu_y + UU_x - uv_y + Uu_x + Uv_y$$

$$= \frac{\partial}{\partial x}(uU - u^2) + (U-u)U_x + \frac{\partial}{\partial y}(vU + vu)$$

$$\int_0^\infty -\frac{1}{\rho}\tau_y dy = -(\cancelto{0}{\tau_\infty} - \tau_w)/\rho = \frac{\partial}{\partial x}\int_0^\infty u(U-u)dy + U_x\int_0^\infty (U-u)dy + \cancelto{0}{(vU - vu)}\Big|_0^\infty$$

$$\frac{\tau_w}{\rho} = \frac{\partial}{\partial x}\left[U^2\int_0^\infty \frac{u}{U}\left(1-\frac{u}{U}\right)dy + U_x\int_0^\infty (U-u)dy\right] =$$

$$U^2\theta_x + 2UU_x\theta + U_x\delta^*$$

$$\frac{C_f}{2} = \frac{d\theta}{dx} + \left(2\theta+\delta^*\right)\frac{1}{U}\frac{dU}{dx}$$

$$\frac{C_f}{2} = \frac{d\theta}{dx} + \left(2+H\right)\frac{\theta}{U}\frac{dU}{dx},\ H = \frac{\delta^*}{\theta}$$

$$\frac{\tau_w}{\rho U^2} = \frac{1}{2}C_f = \theta_x + \left(2+H\right)\frac{\theta}{U}U_x$$

Historically two approaches for solving the momentum integral equation for specified potential flow U(x):

1. Guessed Profiles
2. Empirical Correlations

Best approach is to use empirical correlations to get integral parameters (δ, δ^*, θ, H, C_f, C_D) after which use these to get velocity profile u/U

Thwaites Method

Multiply momentum integral equation by $\dfrac{U\theta}{\nu}$

$$\frac{\tau_w\theta}{\mu U}=\frac{U\theta}{\nu}\frac{d\theta}{dx}+\frac{\theta^2}{\nu}\frac{dU}{dx}\left(2+H\right)$$

The equation is dimensionless and, LHS and H can be correlated with pressure gradient parameter $\lambda=\frac{\theta^2}{\nu}\frac{dU}{dx}$ as shear and shape-factor correlations

$$\boxed{\begin{aligned}&\frac{\tau_w\theta}{\mu U}=S\left(\lambda\right)=(\lambda+0.09)^{0.62}\\&H=\delta^*/\theta=H\left(\lambda\right)=\sum_{i=0}^{5}a_i(0.25-\lambda)^i\end{aligned}}$$

a_i = (2, 4.14, -83.5, 854, -3337, 4576)

Note

$$\frac{U\theta}{\nu}\frac{d\theta}{dx}=\frac{1}{2}U\frac{d}{dx}\left(\frac{\theta^2}{\vartheta}\right)$$

Substitute above into momentum integral equation

$$S(\lambda)=\frac{1}{2}U\frac{d}{dx}\left(\frac{\theta^2}{\nu}\right)+\lambda\left(2+H\right)$$

$$U\frac{d\left(\lambda/U_x\right)}{dx}=2\left[S-\lambda\left(2+H\right)\lambda\right]=F\left(\lambda\right)$$

$F\left(\lambda\right)=0.45-6\lambda$ based on AFD and EFD

Define $z = \frac{\theta^2}{\nu}$ so that $\lambda = z\frac{dU}{dx}$

$$U\frac{dz}{dx} = 0.45 - 6\lambda = 0.45 - 6z\frac{dU}{dx}$$

$$U\frac{dz}{dx} + 6z\frac{dU}{dx} = 0.45$$

i.e., $\frac{1}{U^5}\frac{d}{dx}\left(zU^6\right) = 0.45$

$$zU^6 = 0.45\int_0^x U^5 dx + C$$

$$\Rightarrow \boxed{\theta^2 = \theta_0^2 + \frac{0.45\nu}{U^6}\int_0^x U^5 dx}$$

$\theta_0(x=0) = 0$ and U(x) known from potential flow solution

Complete solution:

$$\lambda = \lambda(\theta) = \frac{\theta^2}{\nu}\frac{dU}{dx}$$

$$\frac{\tau_w\theta}{\mu U} = S(\lambda)$$

$$\delta^* = \theta H(\lambda)$$

Accuracy: mild p_x $\pm 5\%$ and strong adverse p_x (τ_w near 0) $\pm 15\%$

i. Pohlhausen Velocity Profile:

$$\frac{u}{U} = f(\eta) = a\eta + b\eta^2 + c\eta^3 + d\eta^4 \text{ with } \eta = \frac{y}{\delta}$$

a, b, c, d determined from boundary conditions

1) $y = 0 \rightarrow u = 0,\ u_{yy} = -\frac{U}{\nu}U_x$

2) $y = \delta \rightarrow u = U,\ u_y = 0,\ u_{yy} = 0$

No slip is automatically satisfied.

$$F(\eta) = 2\eta - 2\eta^3 + \eta^4$$

$$G(\eta) = \frac{\eta}{6}(1-\eta)^3$$

$$\rightarrow \frac{u}{U} = F(\eta) + \Lambda G(\eta),\ -12 \le \Lambda \le 12$$

separation

(experiment: $\Lambda_{separation} = -5$)

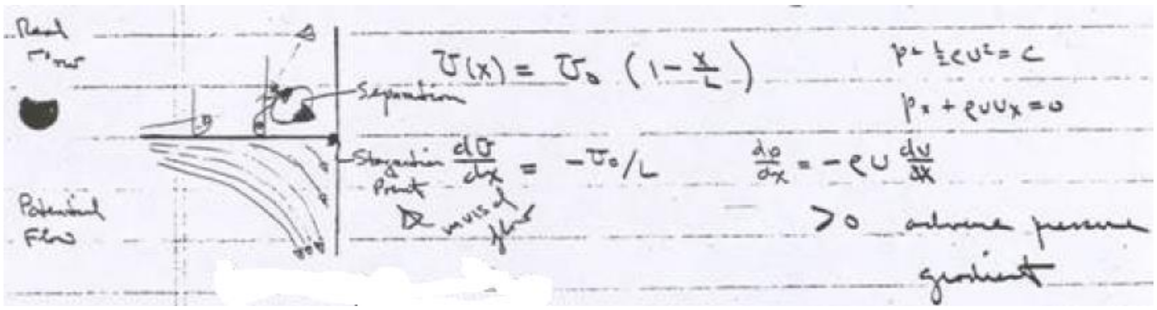
$$\Lambda = \frac{\delta^2}{\nu}\frac{dU}{dx} = -p_x\frac{\delta^2}{\mu U}$$

pressure gradient parameter related to

$$\lambda = \lambda(\Lambda) = \left(\frac{37}{315} - \frac{\Lambda}{945} + \frac{\Lambda^2}{9072}\right)\Lambda$$

Profiles are fairly realistic, except near separation. In guessed profile methods u/U directly used to solve momentum integral equation numerically, but accuracy not as good as empirical correlation methods; therefore, use Thwaites's method to get λ, etc., and then use λ to get Λ and plot u/U.

ii. Howarth linearly decelerating flow (example of exact solution of steady state 2D boundary layer)



Howarth proposed a linearly decelerating external velocity distribution $U(x) = U_0\left(1 - \frac{x}{L}\right)$ as a theoretical model for laminar boundary layer study.

Use Thwaites's method to compute:

a) X_{sep}

b) $C_f\left(\frac{x}{L} = 0.1\right)$

Note $U_x = -U_0/L$

Solution

$$\theta^2 = \frac{0.45\nu}{U_0^6\left(1-\frac{x}{L}\right)^6}\int_0^x U_0^5\left(1-\frac{x}{L}\right)^5 dx = 0.075\frac{\nu L}{U_0}\left[\left(1-\frac{x}{L}\right)^{-6} - 1\right]$$

can be evaluated for given L, Re_L

(Note: $\begin{matrix}\theta = 0 \rightarrow x = 0, \\ \theta = \infty \rightarrow x = L\end{matrix}$)

$$\lambda = \frac{\theta^2}{\nu}\frac{dU}{dx} = -0.075\left[\left(1-\frac{x}{L}\right)^{-6} - 1\right]$$

$$\lambda_{sep} = -0.09 \Rightarrow \frac{X_{sep}}{L} = 0.123$$

3% higher than exact solution =0.1199

$C_f\left(\frac{x}{L} = 0.1\right)$→i.e. just before separation

$$\lambda = -0.0661$$

$$S(\lambda) = 0.099 = \frac{1}{2} C_f \operatorname{Re}_\theta$$

$$C_f = \frac{2(0.099)}{\operatorname{Re}_\theta}$$

Compute Re_θ in terms if Re_L

$$\theta^2 = 0.075 \frac{\nu L}{U_0}\left[(1-0.1)^{-6} - 1\right] = 0.0661 \frac{\vartheta L}{U_0}$$

$$\frac{\theta^2}{L^2} = 0.0661 \frac{\nu L}{U_0} = \frac{0.0661}{\operatorname{Re}_L}$$

$$\frac{\theta}{L} = \frac{0.257}{\operatorname{Re}_L^{1/2}}$$

$$\operatorname{Re}_\theta = \frac{\theta}{L} \operatorname{Re}_L = 0.257 \operatorname{Re}_L^{1/2}$$

$$C_f = \frac{2(0.099)}{0.257} \operatorname{Re}_L^{-1/2} = 0.77 \operatorname{Re}_L^{-1/2}$$

To complete solution must specify Re_L

Consider the complex potential

$$F(z) = \frac{a}{2} z^2 = \frac{a}{2} r^2 e^{2i\theta}$$

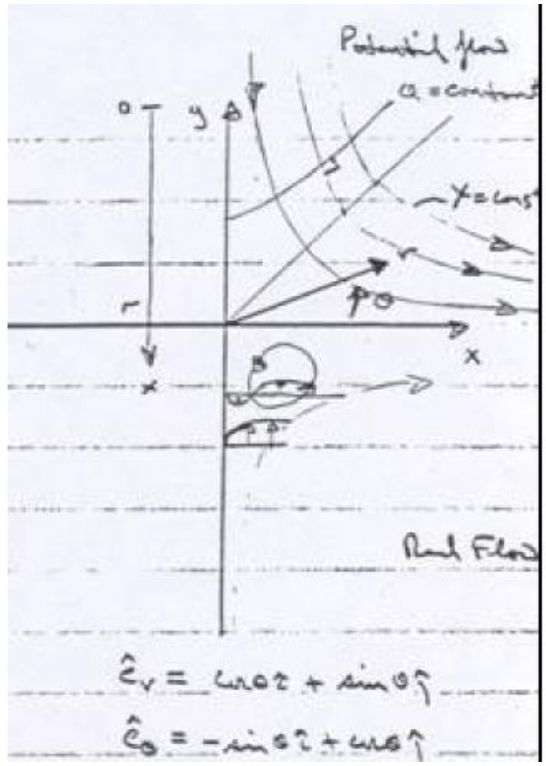
$$\varphi = \mathrm{Re}[F(z)] = \frac{a}{2} r^2 \cos 2\theta$$

$$\psi = \mathrm{Im}[F(z)] = \frac{a}{2} r^2 \sin 2\theta$$

Orthogonal rectangular hyperbolas

φ: asymptotes y = ± x

ψ: asymptotes x=0, y=0



$$\begin{cases} \underline{V} = \nabla \varphi = \varphi_r \hat{e}_r + \frac{1}{r} \varphi_\theta \hat{e}_\theta \\ v_r = ar \cos 2\theta \\ v_\theta = -ar \sin 2\theta \end{cases}$$

$\frac{\pi}{2} \le \theta \le 0$ (flow direction as shown)

$$\underline{V} = v_r (\cos\theta \hat{i} + \sin\theta \hat{j}) + v_\theta (-\sin\theta \hat{i} + \cos\theta \hat{j}) =$$
$$(v_r \cos\theta - v_\theta \sin\theta)\hat{i} + (v_r \sin\theta + v_\theta \cos\theta)\hat{j}$$

Potential flow slips along surface: (consider $\theta = 90^\circ$)

1) determine *a* such that $v_r = U_0$ at r=L, $\theta = 90^\circ$

$v_r = aL \cos(2 \times 90) = U_0 \Rightarrow aL = -U_0$, i.e. $a = -\frac{U_0}{L}$

2) let $U(x) = v_r$ at x=L-r:

$\Rightarrow v_r = a(L - x)\cos(2 \times 90) = U(x)$

$Or: \ U(x) = -a(L - x) = \frac{U_0}{L}(L - x) = U_0 (1 - \frac{x}{L})$

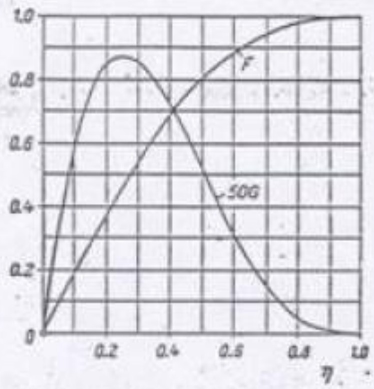


Fig. 10.3. The functions $F(\eta)$ and $G(\eta)$ for the velocity distribution in the boundary layer from eqns. (10.22) and (10.23)

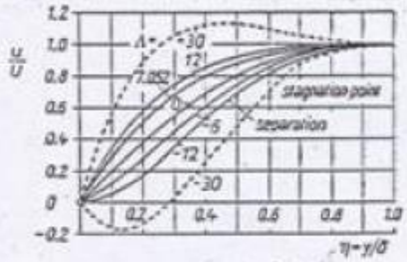


Fig. 10.4. The one-parameter family of velocity profiles from eqn. (10.22)

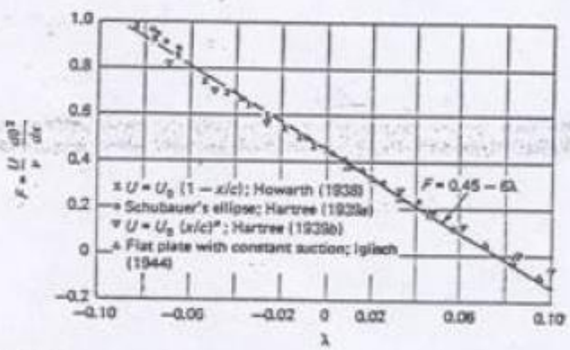


FIGURE 4-23
Empirical correlation of the boundary-layer function $F(\lambda)$ in Eq. (4-189). [*After Thwaites (1949).*]

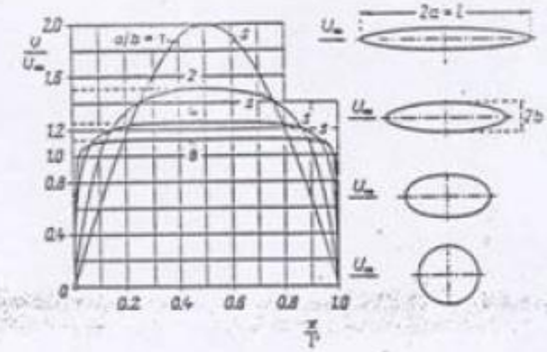


Fig. 10.9. Potential velocity distribution function on elliptical cylinders of slenderness $a/b = 1, 2, 4, 8$, the direction of the stream being parallel to the major axis
S = position of point of separation

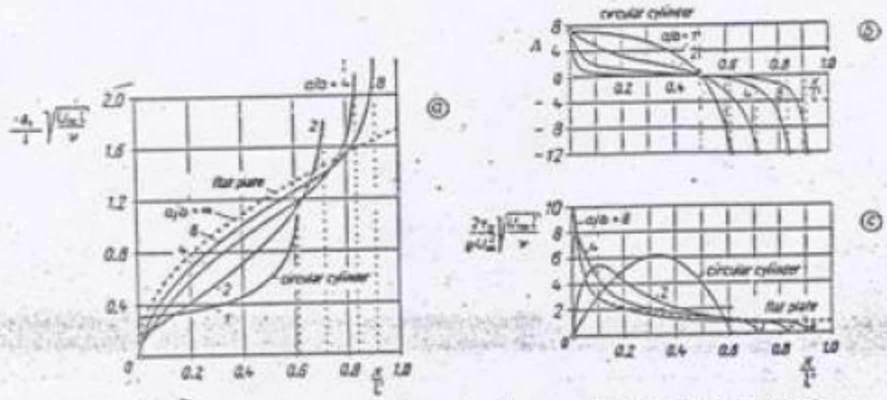


Fig. 10.10. Results of the calculation of boundary layers on elliptical cylinders of slenderness $a/b = 1, 2, 4, 8$, Fig. 10.9. a) displacement thickness of the boundary layer, b) shape factor c) shearing stress at the wall, $2\,l'$ = circumference of the ellipse; $a/b = 1$ circular cylinder; $a/b = \infty$ flat plate

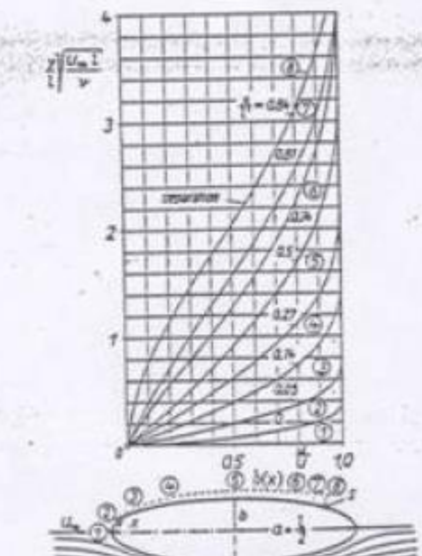


Fig. 10.11. Velocity profiles in the laminar boundary layer on an elliptical cylinder. Ratio of axes $a/b = 4$

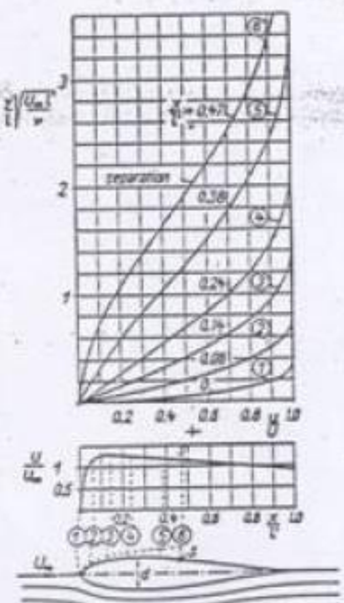


Fig. 10.12. Velocity profiles in the laminar boundary layer and potential velocity function for a Zhukovskii aerofoil J 015 of thickness ratio $d/l = 0{\cdot}15$ at an angle of incidence $\alpha = 0$

Refer back to Table 4-6 (Lec 26) with regard to relative performance of integral methods

<u>Boundary layer with pressure gradient</u>

$$u_x + v_y = 0$$

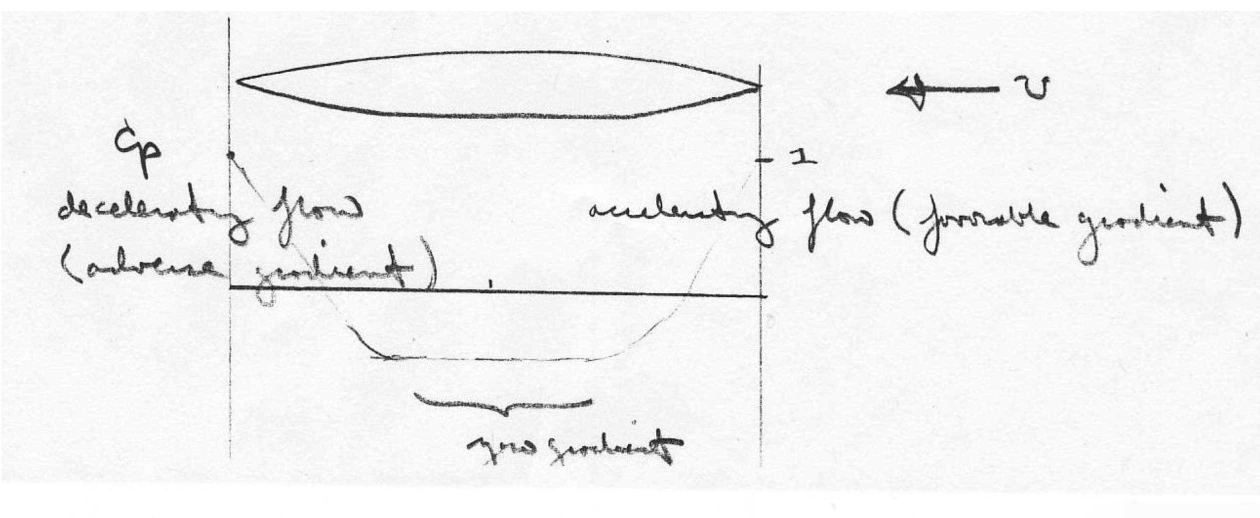
$$uu_x + vu_y = \boxed{-\frac{\partial}{\partial x}(p/\rho)} + \frac{1}{\rho}\frac{\partial \tau}{\partial y}$$

$$\tau = \mu\frac{\partial u}{\partial y} - \rho\overline{u'v'}$$

The pressure gradient term has a large influence on the solution. In particular, adverse pressure gradient (i.e. increasing pressure) can cause flow separation. Recall that the y momentum equation subject to the boundary layer assumptions reduced to

$p_y = 0$ i.e., $p = p_e$ = constant across BL.

That is, pressure (which drives BL equations) is given by external inviscid flow solution which in many cases is also irrotational. Consider a typical inviscid flow solution (chapter 8)



Even without solving the BL equations we can deduce information about the shape of the velocity profiles just by evaluating the BL equations at the wall (y = 0)

$$\mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial p_e}{\partial x}$$

$$where \quad \frac{\partial p_e}{\partial x} = -\rho \mathrm{U_e} \frac{dU_e}{dx}$$

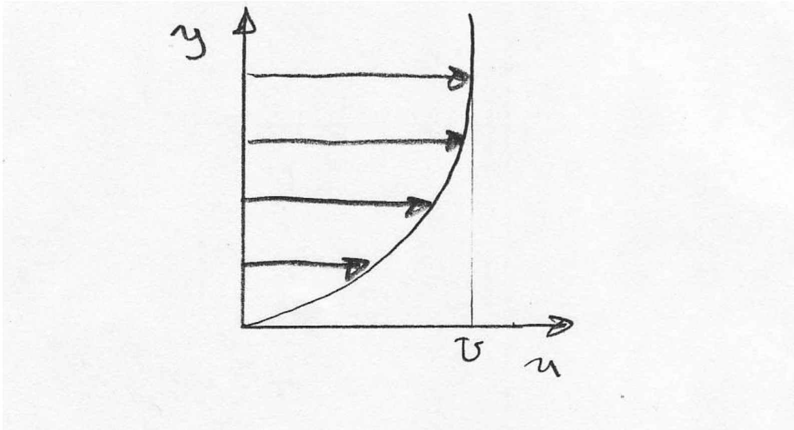
which shows that the curvature of the velocity profile at the wall is related to the pressure gradient.

Effect of Pressure Gradient on Velocity Profiles

Point of inflection: a point where a graph changes between concave upward and concave downward.

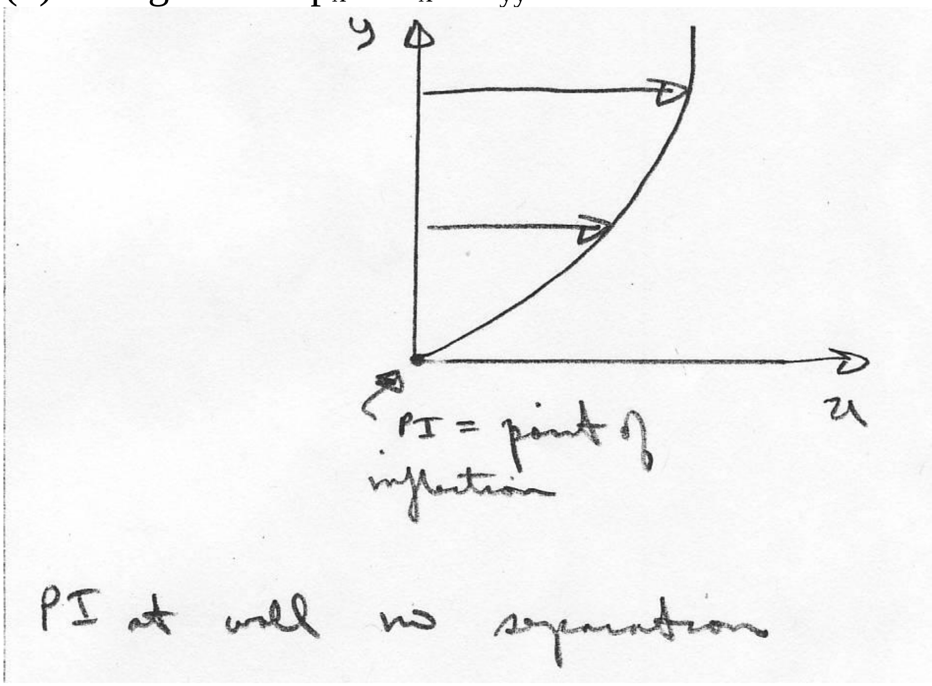
The point of inflection is basically the location where second derivative of u is zero, i.e., $\dfrac{\partial^2 u}{\partial y^2} = 0$

(a) favorable gradient: $p_x<0$, $U_x>0$, $u_{yy}<0$

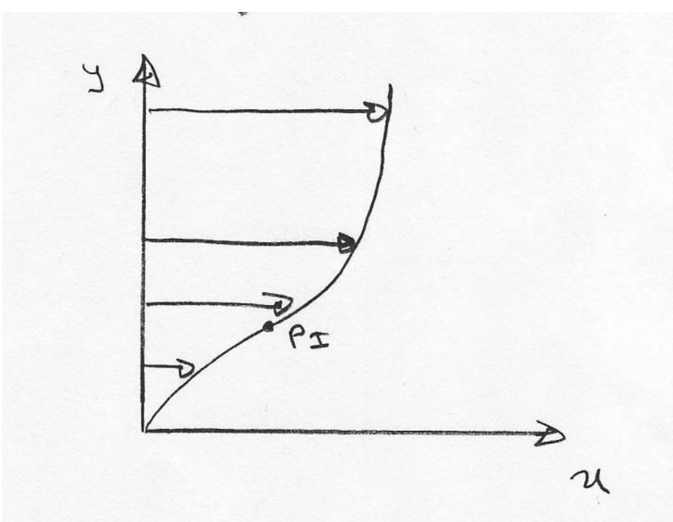


No point of inflection i.e. curvature is negative all across the BL and BL is very resistant to separation. Note $u_{yy}(\delta)<0$ in order for u to merge smoothly with U.

(b) zero gradient: $p_x = U_x = u_{yy} = 0$

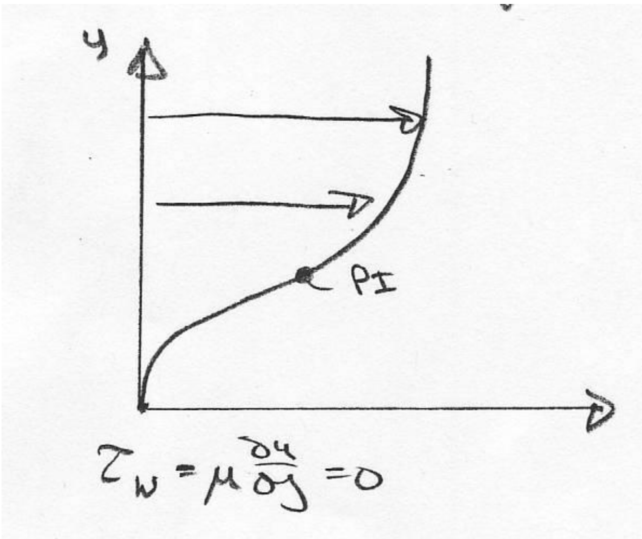


(c) weak adverse gradient: $p_x>0$, $U_x<0$, $u_{yy}>0$



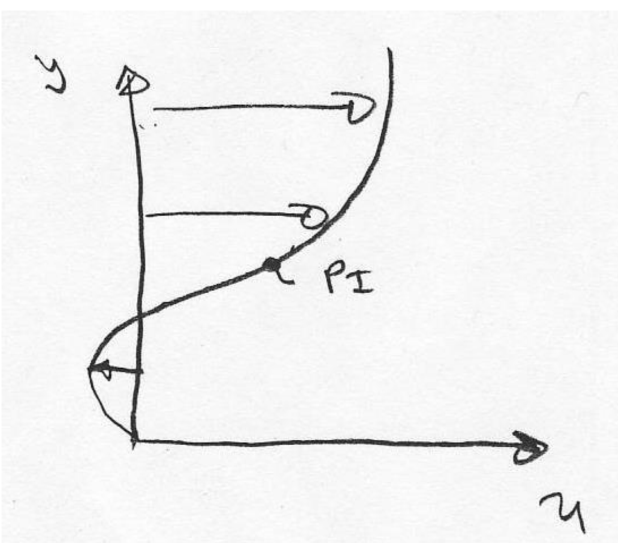
PI in flow, still no separation

(d) critical adverse gradient: $p_x>0$, $U_x<0$, $u_{yy}>0$, $u_y = 0$



PI in flow, incipient separation

(e) excessive adverse gradient: $p_x>0$, $U_x<0$, $u_{yy}>0$, $\underbrace{u_y < 0}_{\tau_w < 0}$



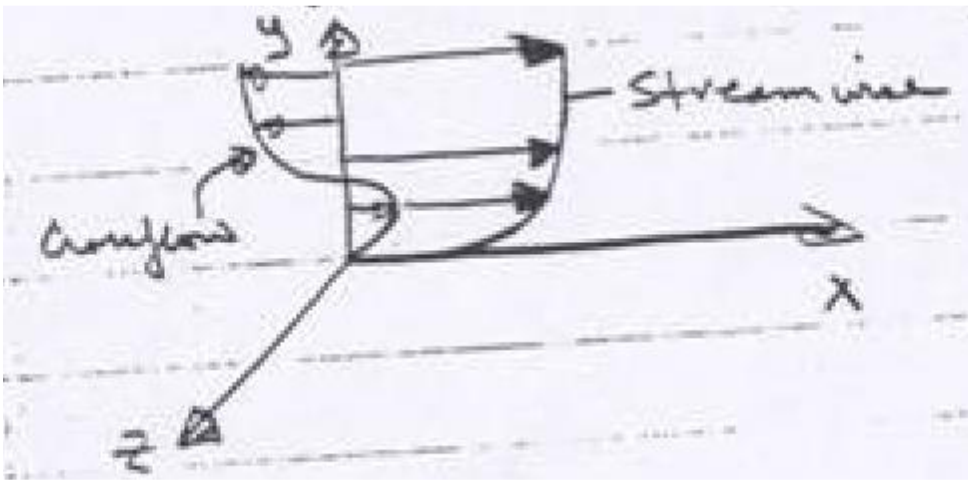
PI in flow, backflow near wall i.e. separated flow region

i.e. main flow breaks away or separates from the wall: large increase in drag and loss of performance:

$H_{separation} = 3.5$ laminar
$= 2.4$ turbulent

3-D Integral methods

Momentum integral methods perform well (i.e. compare well with experimental data) for a large class of both laminar and turbulent 2D flows. However, for ***3D flows they do not***, primarily due to the inability of correlating the cross flow velocity components.



The cross flow is driven by $\frac{\partial p}{\partial z}$, which is imposed on BL from the outer potential flow U(x,z).

3-D boundary layer equations

$$uu_x + vu_y + wu_z = -\frac{\partial}{\partial x}(p/\rho) + \vartheta u_{yy} - \frac{\partial}{\partial y}(\overline{u'v'});$$

$$uw_x + vw_y + ww_z = -\frac{\partial}{\partial z}(p/\rho) + \vartheta w_{yy} - \frac{\partial}{\partial y}(\overline{v'w'});$$

$$u_x + v_y + w_z = 0;$$

$$+ \quad \text{closure equations}$$

Differential methods have been developed for this reason as well as for extensions to more complex and non-thin boundary layer flows.